LAGERSTROM

PROBLEMS IN THE THEORY OF VISCOUS COMPRESSIBLE FLUIDS

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PREFACE

In October 1947 a theoretical investigation of viscous effects in compressible fluids was started at GALCIT* under an ONR** contract. The present report gives a survey of some results obtained during the first year. It is in the nature of a progress report rather than a report on finished research. The project was conceived as a long-range study of the fundamental principles of viscous compressible fluids. The research work discussed in this report represents only the initial phase. The results presented should be further analyzed, extended, and revised. However, because of a growing interest in the subject matter it was decided to publish a report at this stage and present the results for general discussion and criticism.

While the work under the present contract is entirely theoretical, the project grew out of experimental investigations carried out at GALCIT by H. W. Liepmann and his co-workers. In the course of interpreting the experimental results it was felt that existing theory of combined effects of compressibility and viscosity was even qualitatively very unsatisfactory. The authors of this report are very much indebted to Dr. Liepmann not only for discussions of experimental results but also for many theoretical ideas. We also wish to thank Drs. H. F. Bohnenblust, A. Erdelyi, and C. de Prima for valuable discussions of the mathematical aspects of the theory. The manuscript was read by M. D. Van Dyke, who gave useful criticism.

During the summer of 1948 Dr. A. V. Pleijel of the Institute for Advanced Study, Princeton, acted as a consultant on the project and

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achieved essential mathematical progress on many questions. In particular the construction of the fundamental solutions (Appendix D) and the simplified proof of the splitting of the equations are entirely due to him. He also contributed many valuable suggestions about the technique of the Laplace transformation and other mathematical details.

A preliminary summary of some of the results included in this report was presented at the meeting of the Institute for Fluid Mechanics and Heat Transfer in June, 1948, in Pasadena (Ref. 59).

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INTRODUCTION

The present study was suggested by several problems and difficulties that had appeared in previous experimental and theoretical investigations of viscosity effects in compressible fluids. The outstanding problem was the extension of the classical (Prandtl) boundary-layer theory to high-speed flow, especially supersonic flow. In the boundary-layer theory the equations of motion are simplified by assuming that viscous effects are confined to a narrow region close to the wall through which changes are rapid compared to those in the direction of the wall. Then the resulting non-linear equations are studied with the aim of obtaining the flow field in this narrow region or boundary layer. The pressure is usually obtained from the potential or non-viscous flow about the body. Several authors have studied a boundary-layer theory which has the same basic assumptions but which allows for compressibility and heat conduction.* However, in supersonic flow several phenomena are known which show that the basic assumptions of boundary-layer theory do not apply, at least in certain regions.

For example, in the case of uniform supersonic flow at infinity parallel to a semi-infinite flat plate elementary considerations indicate that the viscous effects are not confined to a region close to the plate but that the outer or potential flow field is different from the non-viscous flow. The slowing down of the flow near the plate causes the

*This work was recently summarized by J. A. Lewis (Ref. 20) who gives a complete list of references.

streamlines to turn slightly away from the plate and results in a shock wave standing ahead of the plate followed by an expansion wave. This whole configuration extends into the outer flow field. This shock should be close to the Mach line from the leading edge. In general for supersonic flow certain disturbances spread from the body into the flow field along the Mach lines (characteristics) but up to now little has been said about how viscous disturbances propagate along the Mach lines. This leads naturally to the question of what role is played by the Mach lines or hyperbolic characteristics when viscosity is taken into account.

A consideration of the mathematics of classical boundary-layer theory shows that it is an asymptotic theory for high Reynolds numbers. It cannot be expected to apply for low Reynolds numbers. Low local Reynolds numbers occur at the nose of an object, where the distances involved are small, so that the classical result should not hold near the nose. Another more important case of low Reynolds numbers occurs when the density is very low. For low densities the overall Reynolds numbers may be small, even if the velocity is high. In such cases it seems logical to inquire whether the Mach number is still as important as in non-viscous theory, and in what way the supersonic non-viscous characteristics enter.

Another example of a flow in which the boundary-layer assumptions are invalid is the flow in a region where a shock wave and the classical boundary layer interact. Experimental results (see Ref. 50, 51) show very complicated flow patterns in such regions. A theoretical consideration leads immediately to the negative result that neither simple boundary-layer nor simple shock-wave theory can be applied in this region, since contradictory simplifying assumptions underlie each. Similar difficulties also arise in the theory of the triple shock wave or Mach intersection. It is usually assumed that flow in the neighborhood of the point of intersection can be described by the non-viscous theory of plane shocks and expansion zones but the agreement with experiments is again not very good. The neglected viscous effects may actually be very important in the region of intersection.

Such considerations make it clear that it would be desirable to obtain solutions to some problems of viscous flow without a priori dividing the flow field into regions where the viscosity is considered (as in a boundary layer adjacent to a surface and in a shock wave) and regions in which the non-viscous theory is a good approximation. However due to the complexity of the equations for viscous compressible flow, such as the Navier-Stokes equations (see §1.1), such a task seems hopeless for the exact equations. It seems mandatory that some simplification be made. The method of simplification applied in this report consists in first making very general considerations regarding the problems mentioned and the exact equations in order to discover the typical difficulties and properties associated with them. Then the problems are simplified so that the mathematical problems can be solved but in such a way that one or more of the typical difficulties are preserved or isolated. Or, similarly, we might start out with the very simplest type of viscous wave and gradually increase the complexity of the problems, seeing at each step which feature typical of more complicated problems is introduced.

Following the method of simplification in more detail let us consider the Navier-Stokes equations and boundary conditions. The equations are a set of one first order and four second order partial differential equations for five unknowns (three components of velocity, pressure, and density) linear* in the second order terms but containing non-linear terms of first order (See §1.1). The parameters which enter in the equation are the viscosity μ, heat conduction k, and ratio of specific heats γ. A striking feature is that viscosity ($\mu \neq 0$) introduces second order terms into the momentum equation and raises the order of the system. In this respect the viscous term in the momentum equation differs radically from the conventional first-order friction or dissipation term which represents, for example, the effect of a dashpot on a vibrating system or string. A comparison of conventional friction with viscosity in a fluid is misleading in certain respects. The fact that viscosity raises the order of the system has two immediate consequences not at all in evidence in the theory of ordinary friction; one relates to the characteristics of the system of equations and the other to the boundary conditions to be prescribed in a problem. It is known from the theory of partial differential equations that the essential features of wave propagation are determined by the characteristics and that these characteristic lines or surfaces are determined solely by the highest order terms of the equations (see Ref. 1,

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*Strictly, the equations are non-linear even in the second order terms due to the dependence of μ and k on T. In many practical cases, however, these variations may be omitted.

Vol. II, Chapters 5, 6). Actually the characteristics of the viscous equations change discontinuously as $\mu \to 0$. For small (but finite) μ the characteristics are quite different from those of the non-viscous equations. However, if the non-viscous characteristics are important when $\mu = 0$ the same lines or surfaces must have some meaning when $\mu \neq 0$, or else the non-viscous theory would give a very poor approximation to a real fluid. Similar remarks apply also to the elliptic structure of these equations in the case of steady subsonic flow and this is connected with the question of the significance of the Mach number in viscous flow. The second consequence of the raising of the order of the system for $\mu \neq 0$ as contrasted to $\mu = 0$ is the need for imposing one more boundary condition or a requirement that the solution be continuous. Thus the number of boundary conditions for a problem also changes discontinuously as $\mu \to 0$. For example, the condition of no slip at a wall for $\mu \neq 0$ is usually relaxed when $\mu = 0$.

The difficulties are made specific if an attempt is made to relate the solution for small μ to the solution for $\mu = 0$ by a perturbation procedure. Perturbation methods have been studied in much detail and have found very wide applications in mathematical physics. However, if, as in our case, the order of the equation is lowered, and if one or more boundary or continuity conditions have to be dropped when a parameter (such as μ) is put equal to zero the classical theory does not apply and the perturbation problem is called singular. (See §1.7 for details). The theory of singular perturbations is very little developed.

In accordance with our general program, we shall try to make simplifying assumptions in such a way that the difficulties discussed above remain, but at the same time, the equations become simple enough to make analytical treatment possible. The following are the assumptions made:

i) Heat conduction is neglected.

ii) Disturbances are assumed to be small; the equations are linearized. (However, in Appendices B and C some simple non-linear problems are considered.)

iii) Some simplified and idealized boundary conditions are considered.

The omission of the heat conduction must be justified, for it can be expected that actually the order of magnitude of heat-conduction effects is the same as that of viscous effects. However it can also be expected that their effects are qualitatively similar and this has been checked in several simple cases, although it might not be true if a more exact investigation were wanted. The equations are greatly simplified when the heat conduction is omitted, so that it is felt that a difficulty has been eliminated without altering the nature of the problem.

The linearization may also be justified by saying that it makes mathematical treatment possible and at the same time retains some of the typical features of the more exact problems. The significance of the hyperbolic characteristics and the singular perturbation problems can still be studied for the linearized equations. The main criticism of such a linearization is that in certain regions (for example near a wall) there are large velocity changes due to a condition of no slip at the wall. The linearized equations would strictly apply only locally in certain other regions of the flow. However, if a boundary value problem is to be solved the linearized equations must be taken to apply in the large. This leads to idealized boundary conditions, e. g. for the flat plate a condition of slight slowing down replaces the no slip condition near the wall. From another viewpoint this could be considered as a study of the interaction of the outer part of the boundary layer and the flow at large distances from the plate. It might be remarked that the methods used in the present report might be of interest for rarified gases because of the prescribed slip at the boundary. This idea is not pursued in the present report but will be the subject of further research.

To be more specific the program suggested by the above discussion is as follows:

In Chapter I a system of simplified linearized equations is derived from a more exact system of equations. By comparison of the two systems, it is verified, as claimed above, that some significant properties remain qualitatively unaffected by the linearization. The concepts of longitudinal and transversal waves are introduced and it is proved that any linearized wave (satisfying the simplified system) may be split into a longitudinal and a transversal component.

Chapter 2 deals with basic wave phenomena, corresponding approximately to "sources" or "pulses" in non-viscous fluids. These solutions of the simplified system are of interest in themselves and may also be used as elements in a superposition procedure. First, non-stationary one-dimensional longitudinal waves are discussed ("piston problems"). Rather detailed analytical investigation is possible

(App. A) and from this simple example one may obtain valuable information concerning one question raised above: What role do the non-viscous (hyperbolic) characteristics play in a viscous wave? It is found that they are no longer characteristics in the strict sense of the word so that no discontinuities are propagated along them. On the other hand, under certain conditions depending on local Reynolds number they may be the loci of rapid but still continuous changes. Thus viscosity has the effect of blurring the sharp wave fronts. Next, one-dimensional non-stationary transversal waves are studied. They arise when an infinite flat plate is moved parallel to itself. The solution is independent of compressibility, hence the hyperbolic characteristics mentioned above play no role whatsoever. However, the relevant non-viscous characteristics are instead the stream lines (cf. Ref. 1, Vol. II, p. 375). Again the effect of viscosity is to smear out discontinuities. While for longitudinal waves viscosity has little effect on the boundary conditions it is quite essential for the boundary conditions of the transversal waves. Some non-linear effects are investigated for one-dimensional waves (App. B and C).

In the remainder of Chapter 2 higher-dimensional waves (stationary or non-stationary) are studied. The basic higher-dimensional longitudinal waves may be thought of as generated by cylindrical or spherical pistons or by an irrotational force field concentrated at a point. They behave like one-dimensional longitudinal waves. It is important to study their properties since they will occur as components in more complicated flow phenomena. Simple two-dimensional transversal waves are generated by a rotating cylinder. Such waves are very similar to one-dimensional transversal waves. A limiting case is a wave with rotational symmetry whose vorticity spreads like heat from an instantaneous heat source. Such a wave may also be thought of as generated by a solenoidal force field acting instantaneously at the center. Of particular importance are the waves generated by singular shearing stresses. These waves have both longitudinal and transversal components. They are first discussed from a more intuitive point of view. The singular shearing stress may be thought of as due to an infinitely short flat plate with infinitely strong retarding action. The transversal part is found by considering the vorticity field which turns out to have a dipole singularity. If instead the flat plate is treated by methods analogous to boundary layer theory, only part of the transversal wave is obtained. In either case it is seen that transversal waves are not sufficient to satisfy all conditions of the problem but that longitudinal waves have to be added. These "induced" longitudinal waves will prove to be of fundamental importance also for boundary-value problems. They are part of what might be called the interaction of boundary layer and outer fluid. Finally, the complete solution in any number of dimensions for the flow generated by singular shearing force is deduced by more powerful but less intuitive methods (see Appendix D). The resulting solution is called the fundamental solution of the linearized system of equations. This concept is given a precise mathematical meaning. The solution has the form of a sum of contour integrals. Much work remains to be done in evaluating these integrals and interpreting the results. In the present report some of this work has been done for the two-dimensional stationary case. The interest in this case lies in its relation to stationary flow past a flat plate.

The next step (Chapter 3) is to consider specific boundary-value problems. A natural problem is that of a finite or half-infinite flat plate at zero angle of attack, with a prescribed tangential velocity along the plate. According to conventional boundary-layer theory the flow is undisturbed except for a transversal wave near the plate (boundary layer). This linearized boundary-layer solution is very easy to obtain and is related to the splitting of the system of equations. Of course it should be compared with the solution of the full linearized equations. While this problem is not completely solved preliminary work has been done and it seems possible to obtain considerable information about it with the methods presented. The analysis is partly based on the preceding study of basic wave phenomena. The preliminary result is that in the supersonic case longitudinal waves extend into the outer fluid from near the leading edge in accordance with the natural qualitative guesses. In this example one may use the previously obtained knowledge about the role of the hyperbolic "subcharacteristics" to see to what extent the disturbances near the plate spread into the outer field.

Since linearized equations are studied, there is a still simpler and more basic problem, namely to find the fundamental solution of the equations. A fundamental solution gives us the flow field corresponding to a tangential force applied impulsively at a point in the fluid. In the stationary case it would correspond to having a flat plate of infinitesimal length but with infinite tangential velocity. More complicated flow fields, such as that due to the finite flat plate, may then be obtained by superposition. In this report considerable space has been devoted to studying these fundamental solutions. They have been given for all important cases and analyzed to a certain extent. However, this analysis is far from complete. In particular, superpositions of fundamental solutions have not been investigated. It is to be expected that further research along these lines will yield interesting results.

Actually, the problem of propagation of small disturbances in a viscous fluid, first mentioned by Stokes (Ref. 25) and Kirchhoff (Ref. 26) was studied extensively in France by L. Roy (Ref. 27-33) in 1913-15 and later by Cagniard (Ref. 38-39) and in Belgium by de Backer (Ref. 34-37), (1936-40). In 1942 C. Possio rediscovered the same results, apparently independently (Ref. 42). In particular Roy, Cagniard and Possio all discovered the asymptotic formula (2.27a) which describes the behavior of longitudinal waves at large distances from their origin, and deduced from it the concept of viscous "quasi-wave" which becomes identified with the hyperbolic (Mach) wave in the limit $\nu \to 0$. Meanwhile, transversal waves were being investigated in connection with classical boundary layer theory (Ref. 19) and by Lord Rayleigh (Ref. 43), Wilson (Ref. 44-45), Roberts (Ref. 47), Mache (Ref. 46), Lucas (Ref. 40-41-48) without their connection with longitudinal waves being clearly brought out.

A deeper insight into the physical nature of the problem was shown by P. Duhem (Ref. 22) who first showed in 1900 how both longitudinal and transversal waves followed from the equations of motion. This type of analysis of the linearized equations led Oseen (Ref. 21) to formulate a set of fundamental solutions for an incompressible fluid. In 1928 E. Nordin (23) extended Oseen's work to a compressible fluid and obtained a fundamental solution equivalent to that derived in Appendix D.

The main part of the present report consists of a study of linearized problems. It is desirable also to see, at least qualitatively, what can happen if non-linearity is considered. It is again useful to isolate the various difficulties. For example, non-linearity occurs in the second-order term due to the variable speed of sound. The simplest case where this may be studied is that of a one-dimensional non-stationary longitudinal wave. Furthermore it is known that some essential features of non-linearity are retained if the so-called transonic approximation is made. Heat conduction is still neglected. Appendix B contains a study of such a wave. Another way in which non-linearity is felt is through the quadratic dissipation terms in the energy equation. The simplest example of this is a one-dimensional transversal wave. In the linearized case such a wave does not give rise to any pressure field whereas in the non-linear case the dissipation term requires the existence of a longitudinal pressure wave. In Appendix C it is indicated how this phenomenon may be studied with the aid of an iteration procedure. These are only the very simplest examples of non-linear effects and suggest that non-linearity may not be entirely out of reach. In a way, addition of the viscous term slightly reduces the difficulties due to non-linearity at least if the coefficient of viscosity is considered constant. Non-linearity in lower order terms is not quite as dangerous as in the highest order terms, since standard iteration procedures (such as Picard's) are much less suspect in the former case than in the latter. This points to another application of the linearized solutions discussed in this report as a first step in an iteration procedure.

The most complicated flow pattern discussed in this report is still far too idealized compared to even the simplest high speed viscous flow pattern that can be produced experimentally. However, the idealized flow pattern may still exhibit many of the distinguishing features of the real flow patterns and thus contribute to an understanding of them. Also, in the course of the present investigation many preliminary problems were encountered whose treatment may be of interest. As emphasized before, the present research project is meant as a long range program and it is hoped that the theoretical analysis can be extended. At the same time refinement of experimental technique may make possible experimental studies of simpler flow patterns than have been studied until now.

1. GENERAL PROPERTIES OF EQUATIONS OF VISCOUS FLOW

§1.1 Fundamental Equations

The fundamental equations of this report are obtained from the application of the principles of conservation of mass, momentum, and energy to the hydrodynamical continuum (Ref. 19) or from certain variational principles (Ref. 18). The molecular nature of the fluid is shown by the presence of viscous stresses in the momentum equation, by the equation of state of the perfect gas which is derived from the kinetic theory of gases, and by the existence of a certain internal energy.

The equations may be written

$$\rho \frac{\partial \vec{Q}}{\partial t} + \rho (\vec{Q} \cdot \mathrm{grad}) \vec{Q} = \rho \vec{F} - \mathrm{grad}\, P + \frac{1}{3}\, \mathrm{grad}\, (\mu\, \mathrm{div}\, \vec{Q}) + \mathrm{div}\, (\mu\, \mathrm{grad}\, \vec{Q}) \quad (1.11)$$

conservation of momentum

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, (\rho \vec{Q}) = 0 \quad (1.12)$$

continuity equation

$$\frac{\partial E}{\partial t} + (\vec{Q} \cdot \mathrm{grad})\, E + P \left\{ \frac{\partial (\frac{1}{\rho})}{\partial t} + (\vec{Q} \cdot \mathrm{grad}) \left(\frac{1}{\rho} \right) \right\} = \frac{1}{\rho}\, \mathrm{div}\, (k\, \mathrm{grad}\, T) + \frac{\mu}{\rho} \chi + \mathcal{Q} \quad (1.13)$$

conservation of energy

where χ = dissipation function =

$$\left\{ 2\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\left(\frac{\partial u_2}{\partial x_2}\right)^2 + 2\left(\frac{\partial u_3}{\partial x_3}\right)^2 + \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)^2 + \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)^2 - \frac{2}{3}\left(\mathrm{div}\, \vec{Q}\right)^2 \right\}$$

$$P = \bar{R} \rho T \qquad \text{equation of state} \qquad (1.14)$$

The following notation is used:

time = t, space coordinates $(x_1, x_2, x_3) = \{x_i\} = \vec{x}$

ρ = density

T = temperature

P = pressure

$\vec{Q}$ = velocity vector, (u_1, u_2, u_3) (u, v, w)

$\vec{F}$ = body force vector, per unit mass

$\bar{R}$ = gas constant $= c_p - c_v$

c_p = specific heat at constant pressure, assumed constant

c_v = specific heat at constant volume, assumed constant

E = internal energy $= c_v T$

μ = viscosity $= \mu(T)$ in gas

k = coefficient of heat conduction, assumed $k = k(T)$

$\mathcal{Q}$ = heat added, assumed to be zero, $\mathcal{Q} = 0$

It might be remarked that in the derivation of the Navier-Stokes equations (1.11) two coefficients of viscosity λ and μ enter naturally and it is assumed that $3\lambda + 2\mu = 0$ Ref.(16, 18). If the assumptions are made that stresses are linear functions of rates of strain and the usual conditions of isotropy are used the stress in the x-direction on a face perpendicular to this region may be expressed as,

$$P_{xx} = -P + \lambda \operatorname{div} \vec{Q} + 2\mu \frac{\partial u}{\partial x} \qquad (1.15a)$$

and the shearing forces in the x-direction on the other two faces of the elementary cube are

$$P_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \tag{1.15b}$$

$$P_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \tag{1.15c}$$

In this case (1.11) would be written

$$\rho \frac{\partial \vec{Q}}{\partial t} + \rho(\vec{Q} \cdot \mathrm{grad})\vec{Q} = \rho \vec{F} - \mathrm{grad}\, P + \mathrm{grad}\left[(\lambda+\mu)\,\mathrm{div}\,\vec{Q}\right] + \mathrm{div}(\mu\, \mathrm{grad}\, \vec{Q}) \tag{1.11'}$$

The assumption that $3\lambda + 2\mu = 0$ assures that $P_{xx} + P_{yy} + P_{zz} = 3P$ and shows that no work is done when an element expands but remains similar to itself. Various other approximations for the viscous stresses have been obtained from the kinetic theory and might have to be used in certain applications (Ref. 18). It has also been assumed that the pressure does not depend on the rate of expansion but that the perfect gas law holds.

Since any one of the state variables may be eliminated from the equations (1.11-1.13) by using the equation of state (1.14) and the definition of internal energy, the system (1.11-1.14) should be regarded as a set of five non-linear partial differential equations for the five unknowns, any two of the state variables and the three components of the velocity vector. Each of the equations (1.11) is of second order as is the equation (1.13). However it may be noticed that if k is put equal to zero the order of (1.13) is lowered by one and if

$\mu = 0$ the order of (1.11) is lowered by one. A solution of this system would thus be a set of five functions with certain properties of continuity and differentiability, satisfying the equations in a given domain and assuming prescribed initial and boundary values. Even for the simplest physical problems, no solutions seem to have been given, mainly due to the complexity of the system and the non-linearity of certain second order terms. Also the proper prescription of conditions to give unique solutions for various problems has not been studied in much detail. The underlying structure of this system, as expressed through the characteristics, helps to show what types of problems are mathematically sensible and this structure will be studied in the next section. In this report various approximations will be made and a consideration of the underlying structure is often a help in judging their validity.

§1.2 Characteristics

The existence of real characteristics indicates that the system of partial differential equations is of parabolic or hyperbolic type. Then it can be expected that the type of problem that can be solved and the method of solution should be similar to those used for simpler cases. The characteristic surfaces or lines are found from their definitions as surfaces or lines on which certain derivatives in directions not tangent to the surface cannot be computed from certain prescribed data. The method is described in Ref.1, Vol. II pp. 293-294 in some detail. The characteristics may also be thought of as the loci of possible discontinuities or "wave fronts" which can occur in the solution. According to the general theory, the characteristics are determined only by the highest order terms occuring in the equations and can therefore be expected to change if these terms change. For simplicity, in the computations presented here only one space variable x will be considered, but this does not destroy the essential nature of the system. The entropy I may also be introduced where $T\,dI = dE + p\,d\left(\frac{1}{\rho}\right)$ and we may consider $P = P(I,\rho)$, $T = T(I,\rho)$. Then the equations (1.11-1.13) for a viscous and heat conducting fluid can be written

$$\rho u_t + \rho u u_x + P_I I_x + P_\rho \rho_x - \frac{4}{3}\frac{\partial \nu}{\partial x} = 0 \qquad (1.21a)$$

$$\rho u_x + \rho_t + u\rho_x = 0 \qquad (1.21b)$$

$$T I_t + uT I_x - \frac{\partial \Omega}{\partial x} = \frac{4}{3}\rho\mu\left(\frac{\partial u}{\partial x}\right)^2 \qquad (1.21c)$$

where by definition

$$k\,T_x = k\left(T_I I_x + T_\rho \rho_x\right) = \Omega \qquad (1.21d)$$

and

$$\mu u_x = \nu \qquad (1.21e)$$

If $\Psi(x,t)$ = constant is the equation of the characteristics, the characteristic condition becomes:

$$\begin{vmatrix} \rho\Psi_t + \rho u \Psi_x & P_\rho \Psi_x & P_I \Psi_x & 0 & -\frac{4}{3}\Psi_x \\ \rho\Psi_x & \Psi_t + u\Psi_x & 0 & 0 & 0 \\ 0 & 0 & \rho T(\Psi_t + u\Psi_x) & \Psi_x & 0 \\ 0 & k T_\rho \Psi_x & k T_I \Psi_x & 0 & 0 \\ \mu\Psi_x & 0 & & 0 & 0 \end{vmatrix} = 0 \qquad (1.22)$$

or

$$\frac{4}{3}\mu \Psi_x^2 \begin{vmatrix} \Psi_t + u\Psi_x & 0 & 0 \\ 0 & \rho T(\Psi_t + u\Psi_x) & \Psi_x \\ kT_\rho \Psi_x & kT_I \Psi_x & 0 \end{vmatrix} = 0$$

and finally

$$\frac{4}{3}\mu k T_I \Psi_x^4 (\Psi_t + u\Psi_x) = 0 \tag{1.22'}$$

The characteristics are given by

i) $\Psi_x = 0$ or the lines t = constant which occur as a quadruple characteristic. (1.23a)

ii) $\Psi_t + u\Psi_x = 0$ or $\frac{dx}{dt} = -\frac{\Psi_t}{\Psi_x} = u$, the streamlines (1.23b)

There are thus five sets of characteristics. It should be noted that, as is typical for non-linear equations, the characteristics (ii) (1.23b) are not known in advance but depend on the solution $\vec{Q}(x,t)$.

It is interesting also to consider special cases of simplified fluids. For example, for the fluid with no viscosity, $\mu \equiv 0$ equation (1.21e) must be omitted and the characteristic condition becomes:

$$\Psi_x \begin{vmatrix} \rho(\Psi_t + u\Psi_x) & P_\rho \Psi_x & P_I \Psi_x \\ \rho\Psi_x & \Psi_t + u\Psi_x & 0 \\ 0 & kT_\rho \Psi_x & kT_I \Psi_x \end{vmatrix} = 0 \tag{1.24}$$

or

$$k_\rho T_I \Psi_x^2 \left\{ \Psi_t + u\Psi_x + \sqrt{P_\rho - P_I \frac{T_\rho}{T_I}}\ \Psi_x \right\} \left\{ \Psi_t + u\Psi_x - \sqrt{P_\rho - P_I \frac{T_\rho}{T_I}}\ \Psi_x \right\} = 0 \tag{1.24'}$$

There are now four sets of characteristics:

i) $\Psi_x = 0$ or t = constant occurs twice, as in the ordinary heat equation $u_{xx} = u_t$. (1.25a)

ii) $\Psi_t + (u \pm c_i)\Psi_x = 0$ or $\frac{dx}{dt} = -\frac{\Psi_t}{\Psi_x} = u \pm c_i$ (1.25b)

where

$$c_i = \sqrt{P_\rho - P_I \frac{T_\rho}{T_I}}$$

These characteristics are lines of propagation of disturbances, with the propagation speed c_i relative to the fluid. c_i may be easily evaluated, for

$$P(\rho, I) = \rho^\gamma e^{I/c_v} \tag{1.26a}$$

$$T(\rho, I) = \frac{1}{\bar{R}} \rho^{\gamma-1} e^{I/c_v} \tag{1.26b}$$

and $P_\rho = \frac{\gamma p}{\rho} = c_a^2$, the adiabatic speed of sound squared, and $P_I \frac{T_\rho}{T_I} = \frac{(\gamma-1)P}{\rho}$; so that $c_i^2 = \frac{P}{\rho} = \bar{R}T$ and c_i = isothermal speed of sound.

As another special case one may consider $k \equiv 0$ or no heat conduction. In this case equation (1.21d) must be omitted and the characteristic condition reads

$$\frac{4}{3}\mu\rho T \Psi_x^2 (\Psi_t + u\Psi_x)^2 = 0 \tag{1.27}$$

There are again four sets of characteristics:

i) $\Psi_x = 0$ or t = constant which again occurs double (1.28a)

ii) $\Psi_t + u\Psi_x = 0$ or $\frac{dx}{dt} = u$, the streamline which also occurs twice (1.28b)

Finally, as the last interesting case the compressible fluid with no heat conduction or viscosity may be considered, $k \equiv 0$ and $\mu \equiv 0$. The characteristic condition then reads

$$\rho^2 T(\Psi_t + u\,\Psi_x)(\Psi_t + u\,\Psi_x + \sqrt{P_\rho}\,\Psi_x)(\Psi_t + u\,\Psi_x - \sqrt{P_\rho}\,\Psi_x) = 0 \qquad (1.29)$$

There are now only three sets of characteristics.

i) $\Psi_t + u\,\Psi_x = 0$ or $\dfrac{dx}{dt} = u$ the streamlines, which occur only once. (1.210a)

ii) $\Psi_t + (u \pm c_a)\,\Psi_x = 0$ or $\dfrac{dx}{dt} = u \pm c_a$ (1.210b)
where $c_a^{\,2} = P_\rho = \dfrac{\gamma P}{\rho}$ the adiabatic speed of sound.
These characteristics are lines of propagation of disturbance with the velocity c_a relative to the fluid.

The parabolic nature of the problems is seen when either $\mu \neq 0$ or $k \neq 0$ by the occurrence of the lines t = constant as characteristics. These lines indicate that the signal speed of certain disturbances is infinite. This phenomenon is familiar from the theory of the heat equation $u_{xx} = u_t$. In the case when $\mu = k = 0$ the problem is purely hyperbolic and the signal speed is finite. It is also possible to discuss what discontinuities can exist and to derive laws for the propagation of such discontinuities along the characteristics. It is especially interesting to notice that the characteristics change discontinuously as μ or $k \rightarrow 0$. For small values of μ the "undiscovered" characteristics of $\mu = 0$ must play an important role in the solution and this particular idea will be discussed in some detail later.

§1.3 Linearization of the Equations of Motion

Due to the difficulty of handling the non-linear equations these are not considered in any detail. Instead, in order to gain insight into the nature of the solutions a set of linearized equations will be considered, for which some specific problems can be solved. The linearization can be effected by considering the flow field to differ only slightly from a given flow and to neglect the squares of such differences whenever they appear. Another procedure, equivalent in the first approximation, is to expand the solution in powers of a small parameter ϵ. If $\mathcal{S}(x_i, t)$ is a solution, let

$$\mathcal{S}(x,y,z,t) = \mathcal{S}_0(x,y,z,t) + \epsilon\, \mathcal{S}_1(x,y,z,t) + \epsilon^2 \mathcal{S}_2(x,y,z,t) + \dots$$

where $\mathcal{S}_0(x_i, t)$ is the known basic flow and $\mathcal{S}_1, \mathcal{S}_2 \dots$ are $O(1)$. In given problems the parameter ϵ can often be assigned a physical meaning and the boundary conditions may also be expanded in terms of this parameter. By equating to zero the coefficients of various powers of ϵ which occur in the equation, a set of linear equations is obtained.

For our purposes it is sufficient to consider as the basic flow a steady uniform flow parallel to the x-axis with the velocity U, the velocity at infinity, and constant pressure P_0, temperature T_0, and density ρ_0*. The solution may be expanded as

$$\vec{Q} = U\,\vec{i} + \epsilon\,\vec{q} + \dots \tag{1.31a}$$

$$P = P_0\,(1 + \epsilon p + \dots) \tag{1.31b}$$

$$T = T_0\,(1 + \epsilon\,\Theta + \dots) \tag{1.31c}$$

$$\rho = \rho_0\,(1 + \epsilon s + \dots) \tag{1.31d}$$

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*A particular case is $U=0$. Conversely, the general case $U \neq 0$ may be reduced to the case $U=0$ by a Galilean transformation (§1.4)

It is also assumed that

$$\mu = \mu_0 + \epsilon \mu_1 \quad , \quad \mu_0 = \mu_0 (T_0) \tag{1.31e}$$

$$k = k_0 + \epsilon k_1 \quad , \quad k_0 = k_0 (T_0) \tag{1.31f}$$

$$\vec{F} = - \epsilon \vec{X} \tag{1.31g}$$

From the equation of state (1.14)

$$p = \Theta + s \tag{1.32}$$

s is usually called the condensation. The equations in ϵ then read

$$\rho_0 \frac{\partial \vec{q}}{\partial t} + \rho_0 U \frac{\partial \vec{q}}{\partial x} = - \rho_0 \vec{X} - P_0 \text{ grad } p + \frac{\mu_0}{3} \text{ grad div } \vec{q} + \mu_0 \Delta \vec{q} \tag{1.33}$$

$$s_t + \text{div } \vec{q} = 0 \tag{1.34}$$

$$c_v T_0 (\Theta_t + U \Theta_x) - \frac{P_0}{\rho_0} (s_t + U s_x) = k_0 T_0 \Delta \Theta \tag{1.35}$$

With the aid of the linearized equation of state (1.32) Θ may be eliminated in the left hand side of the energy equation (1.35):

$$p_t + U p_x - \gamma (s_t + U s_x) = \frac{k_0}{c_v} \Delta (p - s) \tag{1.35'}$$

For the special case of a fluid with no heat conduction $(k_0 = 0)$ this becomes

$$p_t + U p_x - \gamma (s_t + U s_x) = 0 \tag{1.35''}$$

which is the same as the equation of linearized isentropic flow. The dissipation, which enters only in the squared terms, has no effect. (1.35'') implies that

$$p - \gamma s = f (x - U t) = 0 \tag{1.36}$$

$$p = \gamma s \tag{1.36'}$$

if the disturbances are initially zero. Thus for $k_0 = 0$, the pressure may be eliminated in (1.33).

If U is put equal to zero, the following equations result:

$$\frac{\partial \vec{q}}{\partial t} + c^2 \operatorname{grad} s = + \vec{X} + \frac{\nu}{3} \operatorname{grad} \operatorname{div} \vec{q} + \nu \Delta \vec{q} \tag{1.37a}$$

$$s_t + \operatorname{div} \vec{q} = 0 \tag{1.37b}$$

where $c = \sqrt{\gamma \frac{p_0}{\rho_0}}$, the adiabatic speed of sound in the undisturbed flow, $\nu = \frac{\mu_0}{\rho_0}$, the kinematic viscosity in the undisturbed flow. The first equation may be modified by using the vector identity

$$\Delta = \operatorname{grad} \operatorname{div} - \operatorname{curl} \operatorname{curl} \tag{1.38}$$

From the system (1.37) an equation for $\vec{q}$ alone is immediately derived. Then:

$$\frac{\partial^2 \vec{q}}{\partial t^2} - c^2 \operatorname{grad} \operatorname{div} \vec{q} = + \frac{\partial \vec{X}}{\partial t} + \frac{\nu}{3} \operatorname{grad} \operatorname{div} \frac{\partial \vec{q}}{\partial t} + \nu \Delta \frac{\partial \vec{q}}{\partial t} \tag{1.39}$$

For $U \neq 0$ the system (1.42) would have resulted. In §1.4 it will be seen, however, that (1.42) follows immediately from (1.37) and thus, is no more general.

The main part of this report will be devoted to a study of (1.37) and special cases thereof. In the examples studied, $\vec{X}$ will be zero except at some singular line or point.

<u>Vorticity</u>. The vorticity field $\vec{\omega}$ of a given flow field is defined by

$$\vec{\omega} = \operatorname{curl} \vec{q} \tag{1.310}$$

A linearized equation for vorticity is derived by applying the operator curl to (1.37a):

$$\frac{\partial \vec{\omega}}{\partial t} = \nu \Delta \vec{\omega} + \operatorname{curl} \vec{X} \tag{1.311a}$$

In addition it follows of course from (1.310) that

$$\operatorname{div} \vec{\omega} = 0 \tag{1.311b}$$

Equations (1.311) characterize the vorticity waves as transversal as defined below (1.53). Notice in particular that, within the linearized theory the propagation of vorticity is independent of compressibility.

The problem of determining a flow field from its vorticity is, however, considerably complicated by compressibility. Some remarks will be made about this problem later.

§1.4 Galilean Transformation. Stationary Flow.

In system (1.37) it is assumed that the velocity vanishes at infinity. In many problems, particularly those involving moving bodies, it is convenient to introduce a system of coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ which moves with respect to the original system. If the object moves in the direction of the negative x-axis with the speed U, it is made stationary by introducing the Galilean transformation

$$\bar{x} = x + Ut \;, \quad \bar{y} = y \,, \quad \bar{z} = z \,, \quad \bar{t} = t \tag{1.41}$$

While in this system the object is stationary, the flow at infinity will have a velocity $U\vec{i}$ and $\vec{q}$ which was the full velocity vector in the original system, now becomes a perturbation velocity, the complete velocity being $\vec{Q} = U\vec{i} + \vec{q}$. In transforming the differential equation one simply has to replace $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial \bar{t}} + U\frac{\partial}{\partial \bar{x}}$, and $\frac{\partial}{\partial x_i}$ by $\frac{\partial}{\partial \bar{x}_i}$. Thus one obtains (using unbarred letters again for the transformed coordinates)

$$\vec{q}_t + U\vec{q}_x + c^2 \operatorname{grad} s = \vec{X} + \frac{4}{3}\nu \operatorname{grad} \operatorname{div} \vec{q} - \nu \operatorname{curl} \operatorname{curl} \vec{q} \tag{1.42a}$$

$$s_t + Us_x + \operatorname{div} \vec{q} = 0 \tag{1.42b}$$

$$\text{Velocity} = \quad \vec{Q} = U\vec{i} + \vec{q} \tag{1.42c}$$

These equations include (1.37) as a special case $(U=0)$. However, their generality is only fictitious since they are obtained from (1.37) by a simple coordinate transformation. In the incompressible case they reduce to the Oseen equations which have been extensively studied by Oseen and others (cf. in particular Ref. 21). The compressible case has been treated in (Ref. 23). All references given after (Ref. 25) belong here too, since the equations are essentially the same. In Oseen's presentation (Ref. 21) equations are derived directly by considering perturbations of uniform flow, rather than by a coordinate transformation of (1.37). This procedure was explained in § 1.3.

The stationary case is of course obtained by assuming all time derivatives to vanish. In other words the equations are obtained from (1.42) by assuming $\frac{\partial}{\partial t} = 0$ or from (1.37) by replacing $\frac{\partial}{\partial t}$ by $U\frac{\partial}{\partial x}$.

Linearized Equations for Stationary Viscous Flow

$$U\vec{q}_x + c^2 \operatorname{grad} s = \vec{X} + \frac{4}{3}\nu \operatorname{grad} \operatorname{div} \vec{q} - \nu \operatorname{curl} \operatorname{curl} \vec{q} \tag{1.43a}$$

$$Us_x + \operatorname{div} \vec{q} = 0 \tag{1.43b}$$

$$\text{Velocity} = \quad \vec{Q} = U\vec{i} + \vec{q} \tag{1.43c}$$

Thus in a sense the equations for stationary flow are a special case of the equations for non-stationary flow with zero velocity at infinity and it is to be expected that solutions to the former may be obtained from solutions of the latter by a method of descent (cf. §2.5). Especially in deriving general theorems (like e.g. theorem about "splitting"; see below) it is very convenient to remember that (1.42) includes (1.37) and (1.43) as special cases. In other cases, e.g. in deriving the fundamental solutions for (1.37) (see Appendix D) it is more convenient to work with (1.37) directly than to use a method of descent.

<u>Vorticity</u>. The equations for propagation of vorticity $\vec{\omega} = \operatorname{curl} \vec{q}$ are easily derived either from (1.311) by a Galilean transformation or from (1.42) or (1.43). They are

$$\frac{\partial \vec{\omega}}{\partial t} + U \frac{\partial \vec{\omega}}{\partial x} = \nu \Delta \vec{\omega} + \operatorname{curl} \vec{X} \tag{1.44}$$

or, in the stationary case,

$$U \frac{\partial \vec{\omega}}{\partial x} = \nu \Delta \vec{\omega} + \operatorname{curl} \vec{X} \tag{1.45}$$

In all cases (1.311b) is still valid: $\operatorname{div} \vec{\omega} = 0$

§1.5 Longitudinal and Transversal Waves. Splitting of the Linearized Equations.

The basic system of linearized equations for a viscous fluid without heat conduction is (1.37). The main part of the present report will be devoted to a study of waves which obey this system. A distinction between two types of waves, longitudinal and transversal, is of fundamental importance. This is emphasized by a theorem that every wave satisfying (1.37) may be represented uniquely as the sum of a longitudinal wave and a transversal wave. This representation will be referred to as a splitting of the wave. A similar splitting of incompressible waves has been discussed by Lamb and Oseen (ref. 16, 21). For the compressible case it has been discussed and applied in Ref. 53. The proof of the possibility of splitting given below is due to A. Pleijel. A more general discussion, applicable to many similar systems of equations is given by Duhem in Ref. 22. Only the non-stationary case will be treated. As pointed out in § 1.4 this includes the stationary case as a special case.

Longitudinal waves: A solution of (1.37) is said to be a longitudinal wave if the velocity field is irrotational: $\operatorname{curl} \vec{q} = 0$. Because of this condition the viscous term $\nu \operatorname{curl} \operatorname{curl} \vec{q}$ drops out of the momentum equation. The remaining term $\frac{4}{3} \nu \operatorname{grad} \operatorname{div} \vec{q}$ may then also be written $\frac{4}{3} \nu \Delta \vec{q}$ by using the operational identity $\operatorname{curl} \operatorname{curl} \equiv \operatorname{grad} \operatorname{div} - \Delta$. Furthermore the impressed force field must be irrotational: $\operatorname{curl} \vec{X} = 0$. If this were not the case a contradiction could be derived by performing the operation curl on both sides of the momentum equation (1.11). Therefore the Equations for Longitudinal Waves are:

$$\vec{q}_t + c^2 \operatorname{grad} s = \frac{4}{3} \nu \operatorname{grad} \operatorname{div} \vec{q} + \vec{X} \equiv \frac{4}{3} \nu \Delta \vec{q} + \vec{X} \tag{1.51a}$$

$$s_t + \operatorname{div} \vec{q} = 0 \tag{1.51b}$$

$$\operatorname{curl} \vec{q} = 0 \tag{1.51c}$$

$$\operatorname{curl} \vec{X} = 0 \tag{1.51d}$$

Thus a longitudinal wave involves a variation in density and hence pressure. It resembles an ordinary non-viscous acoustic wave except for the effect of the term $\frac{4}{3} \nu \operatorname{grad} \operatorname{div} \vec{q}$. Since a longitudinal wave is irrotational there exists a scalar velocity potential $\varphi(x, y, z, t)$ satisfying equations similar to (1.51):

$$\varphi_t + c^2 s - \frac{4}{3}\nu\Delta\varphi - \Xi = 0 \tag{1.52a}$$

$$s_t + \Delta\varphi = 0 \tag{1.52b}$$

$$\vec{q} = \operatorname{grad}\varphi \tag{1.52c}$$

$$\vec{X} = \operatorname{grad}\Xi \tag{1.52d}$$

If s is eliminated, one obtains

$$\varphi_{tt} - c^2\Delta\varphi = \frac{4}{3}\nu\Delta\varphi_t \tag{1.52e}$$

These equations are easily derived from (1.51) if it is assumed that the flow field and the force field vanish sufficiently strongly at infinity.

<u>Transversal Waves:</u> A wave is said to be transversal if its velocity field is solenoidal, i.e. $\operatorname{div}\vec{q} = 0$. From the continuity equation it follows that $s_t = 0$. We shall further postulate that actually $s = 0$. It then follows from the momentum equation that $\operatorname{div}\vec{X} = 0$. The term $\frac{4}{3}\nu\operatorname{grad}\operatorname{div}\vec{q}$ drops out of the momentum equation and the remaining viscous term $-\nu\operatorname{curl}\operatorname{curl}\vec{q}$ may be written $\nu\Delta\vec{q}$. Thus the <u>Equations for Transversal Waves</u> are:

$$\vec{q}_t = -\nu\operatorname{curl}\operatorname{curl}\vec{q} + \vec{X} \equiv \nu\Delta\vec{q} + \vec{X} \tag{1.53a}$$

$$\operatorname{div}\vec{q} = 0 \tag{1.53b}$$

$$\operatorname{div}\vec{X} = 0 \tag{1.53c}$$

$$s = 0 \tag{1.53d}$$

Thus the transversal waves are independent of compressibility, and bear no resemblance to non-viscous acoustic waves. On the other hand, the equations for vorticity of any linearized viscous wave (1.311) are the same as the equations (1.53) above (except of course for the equation

$s=0$). For this reason vorticity waves will sometimes be called transversal.

Since the flow field is solenoidal there exists a vector potential $\vec{A}$ such that $\vec{q} = \operatorname{curl} \vec{A}$. There is also a force potential $\vec{B}$ such that $\vec{X} = \operatorname{curl} \vec{B}$. In two dimensions the vector potential $\vec{A}$ corresponds to the stream function Ψ (which may be thought of as a vector perpendicular to the plane of flow). (For full generality a skew symmetic tensor should be used instead of the vector $\vec{A}$). The following equation is satisfied:

$$\operatorname{curl} \left(\vec{A}_t + \nu \operatorname{curl} \operatorname{curl} \vec{A} - \vec{B} \right) = 0 \tag{1.54a}$$

One may always choose $\vec{A}$ such that

$$\operatorname{div} \vec{A} = 0 \tag{1.54b}$$

However, since in general it is not true that the expression inside the parenthesis in (1.54d) is zero this equation will be of little use.

With the aid of these concepts one may now formulate the

<u>Theorem of Splitting:</u> A wave $(\vec{q}, s)$ satisfying equations (1.37) may be expressed as the sum of a longitudinal wave $(\vec{q}_1, s)$ and a transversal wave $(\vec{q}_2)$. Thus $(\vec{q}_1, s)$ satisfies (1.51), $(\vec{q}_2)$ satisfies (1.53) and $\vec{q} = \vec{q}_1 + \vec{q}_2$. (1.55)

<u>Proof.</u> It will be assumed below that the force $\vec{X}=0$. The extension of the proof to the case $\vec{X} \neq 0$ is easy. A theorem of vector analysis states that any vector field with suitable properties of continuity can be expressed in infinitely many ways as the sum of an irrotational vector field and a solenoidal vector field (see e.g. Ref. 15, p. 37) Thus let

$$\vec{q} = \vec{w}_1 (x_i, t) + \vec{w}_2 (x_i, t) \tag{a}$$

where

$$\operatorname{curl} \vec{w}_1 = 0 \qquad \operatorname{div} \vec{w}_2 = 0 \tag{b}$$

A vector field $\vec{h}$ which is both irrotational and solenoidal is called harmonic. It is clear that if $(\vec{w}_1, \vec{w}_2)$ is a decomposition satisfying (a) and (b) so is $(\vec{w}_1 + \vec{h}, \vec{w}_2 - \vec{h})$, and that conversely, the decomposition is unique up to a harmonic field. If $\vec{q}$ is a solution of (1.37), then $\vec{w}_1$ and $\vec{w}_2$ satisfy the following relationships, derived from (1.37a) and (1.37b), and (b)

$$\frac{4}{3}\nu \operatorname{grad} \operatorname{div} \vec{w}_1 - c^2 \operatorname{grad} s - \frac{\partial \vec{w}_1}{\partial t} = \frac{\partial \vec{w}_2}{\partial t} + \nu \operatorname{curl} \operatorname{curl} \vec{w}_2 \tag{c}$$

or

$$\frac{4}{3}\nu \Delta \vec{w}_1 - c^2 \operatorname{grad} s - \frac{\partial \vec{w}_1}{\partial t} = \frac{\partial \vec{w}_2}{\partial t} - \nu \Delta \vec{w}_2 \tag{d}$$

$$s_t + \operatorname{div} \vec{w}_1 = 0 \tag{e}$$

The general aim is now to separate (c) into two parts each equal to zero, one containing the irrotational field $\vec{w}_1$, and the other the field $\vec{w}_2$. This will be done by adding a suitable harmonic function to $\vec{w}_1$ and subtracting it from $\vec{w}_2$. It will first be shown that there always exists a harmonic function $\vec{h}$, depending on $\vec{w}_1$, such that the functions

$$\vec{q}_1 = \vec{w}_1 + \vec{h} \qquad \vec{q}_2 = \vec{w}_2 - \vec{h} \tag{f}$$

meet the requirements. If $\vec{h}$ is defined by

$$\begin{aligned} \vec{h} &= \int_t^\infty \left\{ \frac{\partial \vec{w}_1}{\partial \tau} + c^2 \operatorname{grad} s - \frac{4}{3}\nu \Delta \vec{w}_1 \right\} d\tau \\ &= \int_t^\infty \left(\nu \Delta \vec{w}_2 - \frac{\partial \vec{w}_2}{\partial \tau} \right) d\tau \end{aligned} \tag{g}$$

Then

$$\operatorname{curl} \vec{h} = 0 \quad \text{for} \quad \operatorname{curl} \vec{h} \sim \operatorname{curl} \vec{w}_1 = 0 \tag{h}$$

$$\operatorname{div} \vec{h} = 0 \quad \text{for} \quad \operatorname{div} \vec{h} \sim \operatorname{div} \vec{w}_2 = 0 \tag{h'}$$

It then follows from (a), (b), and (e) that

$$\vec{q} = \vec{q}_1 + \vec{q}_2 \tag{i}$$

where

$$\operatorname{curl} \vec{q}_1 = 0 \tag{j}$$

$$\operatorname{div} \vec{q}_2 = 0 \tag{j'}$$

Since $\vec{q}$ is a solution of (1.37) $\vec{q}_1$ and $\vec{q}_2$ must satisfy

$$\frac{4}{3}\nu\Delta\vec{q}_1 - c^2 \operatorname{grad} s - \frac{\partial \vec{q}_1}{\partial t} = \frac{\partial \vec{q}_2}{\partial t} - \nu\Delta\vec{q}_2 \tag{k}$$

$$s_t + \operatorname{div} \vec{q}_1 = 0 \tag{l}$$

Introducing (h') in the right hand side of (k)

$$\frac{4}{3}\nu\Delta\vec{q}_1 - c^2 \operatorname{grad} s - \frac{\partial \vec{q}_1}{\partial t} = \frac{\partial \vec{w}_2}{\partial t} - \nu\Delta\vec{w}_2 - \frac{\partial \vec{h}}{\partial t}$$

$$= \frac{\partial \vec{w}_2}{\partial t} - \nu\Delta\vec{w}_2 - \left(\frac{\partial \vec{w}_2}{\partial t} - \nu\Delta\vec{w}_2\right) = 0 \tag{m}$$

Summarizing, the system (1.37) is thus split into two parts

$$\frac{4}{3}\nu\Delta\vec{q}_1 - c^2 \operatorname{grad} s - \frac{\partial \vec{q}_1}{\partial t} = 0 \tag{n}$$

$$s_t + \operatorname{div} \vec{q}_1 = 0 \tag{n'}$$

$$\operatorname{curl} \vec{q}_1 = 0 \tag{n''}$$

and

$$\nu\Delta\vec{q}_2 - \frac{\partial \vec{q}_2}{\partial t} = 0 \tag{o}$$

$$\operatorname{div} \vec{q}_2 = 0 \tag{o'}$$

where the total flow field is given by

$$\vec{q} = \vec{q}_1 + \vec{q}_2 \tag{p}$$

and the pressure is found from system (n).

It will next be shown that the decomposition of the flow field given by (1.55), (n), (o) is unique if the values of $\vec{q}_1$ at $t = -\infty$ are specified. That is, regardless of the initial choice of $\vec{w}_1$ and $\vec{w}_2$ the same functions $\vec{q}_1$, $\vec{q}_2$ result. Assume that there are two decompositions as defined in (1.55)

$$\vec{q} = \vec{q}_1 + \vec{q}_2 = \vec{q}_1{}' + \vec{q}_2{}' \tag{q}$$

Then if

$$\vec{h} = \vec{q}_1 - \vec{q}_1{}' \tag{r}$$

it follows from the fact that $\vec{q}_1$ and $\vec{q}_1{}'$ satisfy (n) that $\vec{h}$ is not only harmonic but that also

$$\vec{h}_t = 0 \tag{s}$$

If the assumption is made that $\lim_{t \to -\infty} (\vec{q}_1 - \vec{q}_1{}') = 0$, $\vec{h}$ is zero identically. Hence $\vec{q}_1 \equiv \vec{q}_1{}'$ and $\vec{q}_2 \equiv \vec{q}_2{}'$.

Conversely, it is trivial that if $(\vec{q}_1, s)$ is a solution of (n) and $\vec{q}_2$ of (o) then $(\vec{q}_1 + \vec{q}_2, s)$ satisfies (1.37).

It should be noted however that in any given problem the systems (n) and (o) are not completely independent. They are in fact related in some way through the boundary conditions of the problem, which are in general prescribed for $\vec{q}$ and s. A more detailed discussion of this remark however has to be set aside for a special problem, where the boundary conditions are specific.

Stationary Waves.

The corresponding definitions and theorems for the stationary case follow immediately from the above by the standard methods of putting time variations equal to zero after a suitable Galilean transformation. Thus assuming that the flow at infinity has the magnitude U and the direction of the x-axis and that s vanishes at infinity one obtains the equations for the stationary case by replacing $\frac{\partial}{\partial t}$ by $U\frac{\partial}{\partial x}$:

Equations for Stationary Longitudinal Waves

$$U\vec{q}_x + c^2 \operatorname{grad} s = \frac{4}{3}\nu \operatorname{grad} \operatorname{div} \vec{q} + \vec{X} = \frac{4}{3}\nu\Delta\vec{q} + \vec{X} \qquad (1.56a)$$

$$Us_x + \operatorname{div}\vec{q} = 0 \qquad (1.56b)$$

$$\operatorname{curl}\vec{q} = 0 \qquad (1.56c)$$

$$\operatorname{curl}\vec{X} = 0 \qquad (1.56d)$$

Similarly the equations for the velocity potential are found from (1.52).

$$M^2\Phi_{xx} - \Delta\Phi = \frac{4\nu U}{3c^2}\Delta\Phi_x \qquad (1.56e)$$

$$\vec{q} = \operatorname{grad}\Phi \qquad (1.56f)$$

Equations for Stationary Transversal Waves

$$U\vec{q}_x = -\nu \operatorname{curl} \operatorname{curl}\vec{q} + \vec{X} = \nu\Delta\vec{q} + \vec{X} \qquad (1.57a)$$

$$\operatorname{div}\vec{q} = 0 \qquad (1.57b)$$

$$\operatorname{div}\vec{X} = 0 \qquad (1.57c)$$

$$s = 0 \qquad (1.57d)$$

Theorem about Splitting of Stationary Waves (1.58)

Any wave obeying (1.42) may be decomposed into a sum of a longitudinal and a transversal wave as defined by (1.56) and (1.57).

Boundary-Layer Equations

The only two-dimensional stationary flow pattern considered in this report is that past a flat plate located on the x-axis. This flat plate has zero thickness and is at zero angle of attack. If in those cases the customary simplifications of the Prandtl boundary-layer theory are applied to the linearized equations the following equations result (it is assumed that there are no external forces applied).

$$U u_x = \nu u_{yy} \tag{1.59a}$$

$$u_x + v_y = 0 \tag{1.59b}$$

$$s = 0 \tag{1.59c}$$

The principles of such an approximation are discussed in Ref. 52, Chapter 4.9. This linearized boundary-layer theory is discussed in Ref. 43 and Ref. 19, p. 138.

A comparison with (1.57) shows that the boundary-layer equations are closely related to the equations for transversal waves. If the boundary-layer approximations had been applied to (1.57) directly the term νu_{xx} would have dropped out from (1.57a) and the same equations (1.59) would have resulted.

§1.6 Types of Problems

Two main types of problems for the partial differential equations given are treated in this report. They are "radiation" problems and boundary value problems. (boundary value is taken to include initial value).

For the radiation problems the effect of a singular force or of certain other singularities is studied. In the non-stationary case, it will usually be assumed that the force is applied instantaneously (impulse); for stationary flow, the force has to be stationary. Many of the solutions discussed in the Chapter on basic waves are of this type, in particular, the fundamental solutions.

The type of boundary value problem which is treated depends of course on the type of the partial differential equations occurring. Whenever the real time t enters explicitly (parabolic or hyperbolic type) we can consider two types of problems, namely:

i) Initial value problems: In this case the unknown function and possibly a certain number of derivatives leading out of the surface $t=0, (u_t, u_{tt} \cdots)$ must be prescribed for $t=0$ and then a solution is sought for $t>0$ in a certain domain. A typical example is the problem of finding the temperature in an infinite slab given its initial distribution.

ii) Mixed boundary value problems: In general, for this case, initial conditions as in (i) are prescribed, and an additional condition on the time-axis (or time-like surface) is prescribed. The solution is then sought for $t>0$ in some, usually restricted, domain of space. An example of this type of problem is the one-dimensional "piston problem" as usually understood in aerodynamics. The gas could be taken to be initially at rest and the piston be given a prescribed motion with time. Thus there is a boundary condition at the piston for all time and the solution might be sought in the space on one side of the piston. In a study of mixed boundary value problems it becomes evident that sometimes not as many conditions can be prescribed along the time axis as on the initial surface.

In this report we shall also treat some viscous equations in which the time does not enter (stationary flow). In this case the behavior is like that of an elliptic equation complicated, however, by the partly parabolic nature of the equation and the conditions are prescribed on a boundary enclosing the fluid. Part of the boundary, of course, may be at infinity. The unknown function itself is usually given on the boundary, and on part of the boundary some derivatives must also be prescribed. This is due to the fact that we are now treating equations of order higher than two. An example of a problem like this is a flat plate at zero angle of attack in a uniform flow. It turns out that the velocity must be prescribed at the plate and that at infinity upstream the velocity and its derivative in the flow direction must be given.

In connection with all the above problems, the following question is often of great interest. We are concerned with the dependence of the solution on certain physical parameters occurring in the equations, such as viscosity μ, heat conduction k, etc. This leads naturally to the study of perturbation problems which relate the solution for a certain value usually zero of the parameter to the solution for other values of this parameter.

§1.7 Singular Perturbation Problems

For the concepts discussed in this section, compare Ref. 14, 52 Ch. 4.9, 54. The last paper includes a list of additional references. It can be seen from the Navier-Stokes equations (eq. 1.11) that if either of the parameters μ or k is put equal to zero the order of the system of partial differential equations is lowered by one. Since the number of boundary conditions necessary for a problem (i.e. to guarantee a unique solution) depends on the order of the equations, it is seen that one more boundary condition is needed if μ and k do not equal zero than when either μ or k is equal to zero. The number of boundary conditions necessary thus changes discontinuously as $\mu \rightarrow 0$. If the character of the flow field changes drastically when $\mu \rightarrow 0$, as for example when the boundary condition of no slip is relaxed, it might be thought that the solution for small values of μ is related to the solution for $\mu = 0$ in some special way. Actually this relationship is an example of a singular perturbation problem.

Now, if a certain equation and possibly boundary conditions depend on a parameter, ϵ, a perturbation problem is in general defined as the problem of finding the solution for small values of ϵ, given the solution for $\epsilon = 0$. In general the order of the equations and the number of boundary conditions remains fixed in the procedure, and then the problem is called an ordinary or regular perturbation problem. There are many examples of such problems in mathematical physics and the theory of them has been developed extensively, especially for ordinary differential equations. However, if the order of the equation is lowered when $\epsilon = 0$ and if one or more boundary conditions have to be dropped, the perturbation problem is called singular. The theory of singular perturbation problems is rather incomplete compared with that of regular problems (see Ref. 14).

It should be emphasized that the "boundary conditions" mentioned above sometimes consist of requirements on the continuity of the solution. This will be discussed in § 2.3 and following sections.

The outstanding examples of treatment of singular perturbation problems have thus far occurred in the theory of viscous fluids, at least for partial differential equations. The boundary layer theory of Prandtl can be considered a perturbation procedure for a singular problem. (Ref. 52, Ch. 4.9). The solution in this case depends on certain assumptions which are not proved but are made quite plausible. The work of Oseen and his pupils bears on the singular perturbation problems in another way. It is an attempt to check the perturbation procedure for certain simplified equations for a viscous incompressible fluid. In this work solutions of the simplified equations with $\nu \neq 0$ are compared with those for $\nu = 0$. Outside of these examples the theory, in application to physical problems, has been somewhat neglected.

The special nature of singular perturbation problems has been emphasized in some mathematical papers by Friedrichs and Wasow (Ref. 14, 54). These authors deal with ordinary differential equations and some comparitively simple partial differential equations. Qualitatively, some of their solutions exhibit features which appear in the solutions of the viscous equations. One of their results is that, as $\epsilon \to 0$, the solution of certain second order differential equations $(\epsilon y'' + y' + \dots = 0)$, approaches the solution of the corresponding problem with $\epsilon = 0$, uniformly, except in a zone of rapid readjustment, which is necessitated by the extra boundary condition when $\epsilon > 0$. (A similar statement is true for higher order equations and systems). This zone corresponds to a boundary layer. Sometimes there are also zones of rapid change in the interior which would correspond to shock waves. Similarly the results of Ref. 54 show the analogue of the wake behind a blunt object which has been studied by Oseen.

The authors of this report believe that it is very important to continue the general research on singular perturbation problems and to treat the slightly viscous fluid from this point of view. It should also be noted that other singular perturbation problems can occur in the theory of real fluids. Whenever any parameter in the equation, the heat conduction, viscosity, or mean density is very small (or very large) one should investigate whether putting this parameter equal to zero (or infinity) changes the order of the equation, so that a boundary condition will have to be omitted. In this report the approach of singular perturbation problems has not been used extensively. Some comparisons of solutions for $\nu = 0$ and $\nu > 0$ have, however, been made.

2. BASIC WAVE PHENOMENA

In this chapter simple examples of the various types of waves will be studied. The chapter serves a double purpose. The wave phenomena treated will be comparatively simple instances of solutions of the equations and thus of interest in themselves. Using them as specific examples it will be possible to study some of the problems mentioned in the Introduction. The second purpose is to obtain the basis for constructing more complicated (but also more realistic) solutions by superposition. The solutions will be either so-called fundamental solutions (see below), or related solutions. First some comments will be made on the concept of fundamental solutions.

§2.1 Concept of Fundamental Solutions.

For the general theory of fundamental solutions and for detailed discussion of specific examples the reader is referred to Ref. 1, Vol. II, Chapter 4 (elliptic equations), Ref. 12, Chapter X (heat equation), Ref. 3 (more general parabolic equations), Ref. 21 (equations for viscous incompressible fluids). For our purpose a few comments on the properties of fundamental solutions will suffice. We will consider as a concrete example the fundamental solution of the Poisson equation in two dimensions. This equation may be regarded as the equation for a membrane in equilibrium under a distribution of loads. It is

$$\Delta W = W_{xx} + W_{yy} = -f(x,y) \tag{2.11}$$

where

W = vertical deflection of membrane

f = distribution of external forces

The fundamental solution is familiar from its interpretation as an "influence function," telling what happens at one point due to a certain singular disturbance at another. In its more general meaning however it is the kernel of an integral operator which inverts the differential operator in question.

For example, a fundamental solution of (2.11) is (Ref. 1, Vol. II, p. 242):

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\log r, \quad r^2 = (x-\xi)^2 + (y-\eta)^2 \tag{2.12}$$

The fundamental solution given by (2.12) represents the deflection at the point (x,y) caused by a Dirac force applied at the point (ξ,η).* A Dirac-force function $\delta(x,y;\xi,\eta)$ represents a force distribution that may be thought of as $\lim\limits_{\varepsilon \to 0} \delta_\varepsilon$ where

$$\begin{aligned} \delta_\varepsilon(x,y;\xi,\eta) &= \frac{1}{\pi\varepsilon^2} && \text{for } r < \varepsilon \\ &= 0 && \text{for } r > \varepsilon \end{aligned} \tag{2.13}$$

Thus f vanishes except at $\xi = x,\ y = \eta$. At this point it is infinite but its area integral is unity.

It is implicit in the discussion above that G, regarded as a function of x and y, satisfies the homogeneous equation $\Delta W = 0$ except at $x = \xi,\ y = \eta$. In other examples the right hand side f(x,y) might not signify a force distribution but, for example, a force potential, a

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*If W had been required to vanish along a given boundary, a regular harmonic function should have been added to (2.12). This will give the Green's function for the region in question (Ref. 1, Vol. II, Ch. 4). In the present report we shall mostly be concerned with fundamental solutions that vanish at infinity in a certain way.

distribution of hydrodynamical sources or heat sources. The interpretation then has to be changed accordingly.

The second property has more precise mathematical content, independent of any physical interpretation. It says that

$$\Delta \iint G(x, y ; \xi, \eta) f(\xi, \eta)\, d\xi\, d\eta = -f(x, y) \tag{2.14}$$

Thus, in a way, the integral operator constructed with G as kernel is the inverse of the Laplacian Δ . This second property is of course made plausible by the first property combined with the superposition principle.

Although formally much more complicated, the fundamental solutions of the viscous equations will in principle be of the same nature as the above example. The intuitive interpretation depends of course on the interpretation of the equation. The viscous fundamental solutions will not exhibit the strongly singular behavior of the fundamental solutions of hyperbolic equations (such as the supersonic source) but will in their convergence and regularity properties behave more like the solutions of heat equations.

In a system with many dependent variables the fundamental solution will be a tensor (Ref. 21, p. 22). This may be made plausible in the following way. If we regard the system as a vector equation, we have a vector function $\vec{f}$ instead of f in (2.11). It might be a vector function of three dimensions, as for the actual viscous equations. If $\vec{f}$ is reduced to a Dirac force at ξ, η its direction may still be specified by a unit vector $\vec{e}$. For each $\vec{e}$ we obtain a vector field with a singularity at (ξ, η). The fundamental solution G is then a tensor field depending on ξ and η as parameters such that $G \cdot \vec{e}$ is the vector field generated when the Dirac force at (ξ, η) has the direction $\vec{e}$. It is also clear that if (2.14) is to make sense formally, G has to be an n-by-n tensor when $\vec{f}$ is an n-dimensional vector.

§2.2 One-Dimensional Longitudinal Waves

As a simple example of a longitudinal wave let us consider a typical piston problem as understood in aerodynamics: waves travel in a half-infinite tube bounded at the end by a piston, and the effects of the side walls are neglected (or the piston could be considered an infinite plane) so that only one-dimensional waves are produced. The disturbances produced are assumed to be small so that the linearized equations are good approximations. Heat conduction and external forces are neglected. The fluid is assumed to be initially at rest when the piston starts its motion, sending out waves.

The motion of the piston can be described by a curve in the (x,t) plane: $x = \varepsilon\, g(t)$ say, where the motion is assumed to have small amplitudes, $g(t) = O(1)$. The boundary condition to be satisfied at the piston is that the fluid adheres to the piston and has the same velocity as the piston. Thus, exactly (see §1.3), if $u = u\,[x,t]$:

$$\varepsilon\, u\left[\varepsilon\, g(t), t\right] = \varepsilon\, g'(t) \tag{2.21}$$

and this may be expanded as

$$\varepsilon u\,(0,t) + \varepsilon^2 g(t)\, u_x(0,t) + \ldots = \varepsilon\, g'(t) \tag{2.21'}$$

Thus, for the first approximation, the boundary condition is applied at $x = 0$:

$$u(0,t) = g'(t) \tag{2.22}$$

The other boundary condition specifies damping at $x = \infty$.

The equations which apply are a special case of system (1.37) with $U = 0$, $\vec{X} = 0$ and one space variable x:

$$u_t + c^2 s_x = \frac{4}{3} \nu u_{xx} \tag{2.23a}$$

$$s_t + u_x = 0 \tag{2.23b}$$

Again, $p = \gamma s$ according to (1.36'). In this case, the waves are purely longitudinal, as can be seen by comparing (2.23) and (1.51). If the velocity potential Φ is introduced, (2.23) becomes (cf. 1.52e):

$$\Phi_{tt} - c^2 \Phi_{xx} = \frac{4}{3} \nu \Phi_{xxt} \tag{2.24a}$$

$$s = \frac{1}{c^2} \left(\frac{4}{3} \nu \Phi_{xx} - \Phi_t \right) \tag{2.24b}$$

The velocity u satisfies the same equation as Φ.

For $\nu = 0$, equations (2.23) and (2.24) reduce to the ordinary one-dimensional sound wave equations. Viscosity is sometimes compared to a kind of frictional resistance whose main effect is dissipation of energy. However, this comparison is quite misleading. First it should be observed that when the non-linear terms are left out, the viscous dissipation term χ (see 1.13) is neglected altogether. As a matter of fact, since heat conduction is also neglected, the energy relation is the same as for linearized isentropic flow. Furthermore, in the customary examples of waves in a dissipative medium, the terms which account for the dissipation are of order lower than two, whereas in equation (2.24a) the viscous term is of third order. It is known from the theory of differential equations that this is a very essential difference. It will be seen that because of this, the primary effect of viscosity is dispersive rather than dissipative. The theory of characteristics at least indicates this, and it can be verified in the explicit solutions.

For $\nu \neq 0$, equation (2.24a) is a parabolic third-order equation. In the (x,t) plane the characteristics are given by the lines x = constant and t = constant (cf. §1.2). The first set is of no interest in the present connection. The second set comes from a double root of the characteristic equation. It is exactly the same set that appears in the heat equation, which is the classical example of a second-order parabolic equation. It indicates that the signal velocity of a disturbance is infinite. This fact shows that, just as for the heat equation, the underlying theory is of a statistical nature. Disturbances are propagated by molecules, and in the actual statistical model used there is no upper bound on the possible velocities of the molecules.

However, if ν becomes zero, the characteristics change discontinuously from lines parallel to the x-axis to the ordinary hyperbolic characteristics of the acoustic equation, namely the lines $x \pm ct$ = constant. This means that the signal velocity changes abruptly from infinity to the finite value c. We know from acoustics that these hyperbolic characteristics determine the propagation of disturbances. For $\nu \neq 0$ they are no longer characteristics in the usual sense of the word. In this case, we shall refer to them as subcharacteristics. It seems reasonable that, in some sense, these subcharacteristics should be important even for $\nu \neq 0$. If this were not the case, the non-viscous theory would be a very poor approximation to the theory of real fluids. Thus we encounter here a special example of the general situation mentioned in the Introduction.

To investigate the role of the subcharacteristics, we consider some specific solutions. Assume that the fluid is undisturbed for $t < 0$ and that at $t = 0$ a signal is started by moving the piston. For $\nu = 0$ this velocity wave is propagated unchanged with a speed c. With the aid of standard mathematical techniques such as the Laplace transformation (Ref. 53) or Fourier analysis (Ref. 32, 42) a solution is obtained for equation (2.24).

As a first example, we consider the case when the piston velocity is a step signal. The boundary conditions are then:

condition at the piston
$$\left.\begin{array}{ll} u(0,t) = 0 & t < 0 \\ u(0,t) = u_0 & t > 0 \end{array}\right\} \qquad (2.25a)$$

damping at infinity
$$u(\infty,t) < \infty \qquad (2.25b)$$

initial conditions
$$u(x,0) = u_t(x,0) = 0 \qquad (2.25c)$$

Note that, in the inviscid case, the corresponding boundary conditions are (2.25a) and (2.25c). Condition (2.25b) is also satisfied, but this follows from the character of the solution and (2.25b) is not an independent boundary condition.

The role of the boundary conditions is clearly seen in Appendix A, where the problem formulated above is solved by means of a Laplace transformation. Thus, it is necessary to specify both u and u_t in the initial conditions, because, for the viscous fluid equations as well as for the inviscid ones, the highest order of derivative with respect to time is two. Similarly, the transformed equation in either case is a second-order total differential equation. Boundary condition (2.25b) is automatically satisfied in the case of the inviscid fluid, when the requirement is met that the solution of the transformed equation be a Laplace transform. But, due to the presence of a branch-point in the solution of the transformed equation of a viscous fluid, condition (2.25b) is needed to determine the solution uniquely.

There is thus very little difference between the boundary conditions needed to formulate the problem for a viscous and an inviscid fluid. It will be seen from the discussion of specific solutions that the essential condition which may be added for viscous waves is a continuity requirement on the solution. A similar situation with respect to continuity requirements and boundary conditions holds for longitudinal waves in higher dimensions, as will be seen in § 2.4 and § 2.5. In general, the inviscid fluid has a tangential velocity component along the boundary; in the case of a viscous fluid, a no-slip condition, or some similar condition is then imposed on that tangential component and transversal waves also arise. The boundary conditions for a viscous fluid, then, differ from those applied to an inviscid fluid. However, this added boundary condition may also be regarded as a continuity requirement (see § 2.3).

The solution of problem (2.23) with conditions (2.25) is:

$$u(x,t) = \frac{u_0}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\sigma t} \, e^{-\frac{\sigma x}{\sqrt{c^2+\frac{4}{3}\nu\sigma}}} \frac{d\sigma}{\sigma} \tag{2.26}$$

In formula (2.26), x and t represent the physical coordinates. The formula is obtained from equation (A.9a) by the substitution

$$t' = \frac{c^2 t}{\frac{4}{3}\nu} \qquad x' = \frac{c x}{\frac{4}{3}\nu} \tag{2.27}$$

Without solving equation (2.23), one notices that there is no overall Reynolds number for this problem, but there are two independent local Reynolds numbers, one based on position, and the other on time. These are exactly the combinations x' and t' in (2.27).

It does not seem feasible to evaluate the contour integral (2.26) in closed form. It defines a new function, which, together with related functions, occurs frequently in more complicated problems (cf. §2.4 and Appendix A). But one obtains a qualitative idea of the behavior of this function by studying various asymptotic forms of the integral.

For large values of t, the following formula is useful:

$$\frac{u(x,t)}{u_0} = \frac{1}{2}\,\mathrm{erfc}\,\frac{x-ct}{\sqrt{8/3\,\nu t}} + \mathcal{E}_0 \tag{2.28a}$$

where the remainder term $\mathcal{E}_0$ satisfies the inequality:

$$|\mathcal{E}_0| < \sqrt{\frac{8\nu}{9\pi c^2 t}} + O\left(\frac{1}{\sqrt{\frac{c^2 t}{\frac{4}{3}\nu}}}\right) \tag{2.28b}$$

The complementary error function $erfc$ is discussed in Ref. 12, Appendix II.

Another representation suitable for small values of t is:

$$\frac{u(x,t)}{u_o} = erfc \frac{x}{2\sqrt{\frac{4}{3}\nu t}} + \varepsilon \tag{2.29a}$$

where

$$|\varepsilon| < \frac{2c^2 x\sqrt{t}}{\left(\frac{4}{3}\nu\right)^{\frac{3}{2}}} \tag{2.29b}$$

Before discussing the significance of the formulas for the propagation of a step signal into a viscous fluid, we now give the formulas for the propagation of a pulse (Dirac signal). We assume that the piston starts moving at very high speed, and then stops after a very short time. Boundary condition (2.25a) must then be replaced by:

$$u(o,t) = u_o \ \delta(t) \tag{2.210a}$$

where $\delta(t)$ is the Dirac function defined by:

$$\int_0^{\varepsilon} \delta(t)\, dt = 1 \qquad \text{for any } \varepsilon \tag{2.210b}$$

The solution is again in the form of a contour integral:

$$\frac{u(x,t)}{u_o} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\sigma t}\, e^{-\frac{\sigma x}{\sqrt{c^2+\frac{4}{3}\nu\sigma}}}\, d\sigma \tag{2.211}$$

e asymptotic formula which corresponds to (2.28) is:

$$\frac{u(x,t)}{u_o} = \sqrt{\frac{4\nu}{3\pi c^2 t}}\ e^{-\frac{(x-ct)^2}{8/3\,\nu t}} + \varepsilon_1 \tag{2.212a}$$

where

$$|\varepsilon_1| < \frac{16\nu}{3\sqrt{3}\,\pi c^2 t} + o\left(\frac{\frac{4}{3}\nu}{c^2 t}\right) \tag{2.212b}$$

Let us start the analysis of these formulas by an examination of (2.28) and (2.212), neglecting at first the remainder terms ε_0, ε_1. For the step signal, the solution of the inviscid flow equation is the familiar discontinuous wave front which moves at the velocity c. If u is considered as a function of x for a given value of $t=t_0$, then:

$$\frac{u}{u_0} = 1 \qquad x < ct_0$$
$$= 0 \qquad x > ct_0 \tag{2.213}$$

A comparison of this result with formula (2.28a) shows that viscosity has a "smoothing out" effect. Instead of the discontinuity at $x = ct_0$ there is a rapid continuous transition (see Figure 2.1). The smaller the value of ν, the steeper the wave front; on the other hand, for a fixed value of ν, the wave front becomes less steep as t increases. Thus, viscosity has a dispersive effect which increases with time. Notice also that the signal velocity is infinite: there are disturbances at any value of x. This is consistent with the existence of parabolic characteristics for equation (2.24a). The lines x - ct = constant, which would be characteristics if ν vanished (sub-characteristics), do not describe the propagation of a discontinuity; but, in a sense, the line x = ct is the center of the wave. And it can be shown by carrying out a Fourier analysis of the integral (2.26) that c is actually the group velocity of the disturbance (see Ref. 53). Thus, while the characteristics are associated with the signal velocity of the disturbance, the sub-characteristics, here, determine its group velocity.

It is necessary to add the following remark in connection with the order of magnitude of the remainder terms, such as $\mathcal{E}_o$. For a given value of $t = t_o$, the bound on $|\mathcal{E}_o|$ is determined by the inequality (2.28b). If the parameter x is chosen so that $x - ct_o$ is sufficiently large, then the asymptotic value of $\frac{u}{u_o}$ given by formula (2.28a) may become smaller than the remainder term. This fact seems to invalidate formula (2.28a) as an asymptotic formula.

To clarify that point, examine Figure 2.1. The remainder term $\mathcal{E}_o$ determines a region in the u,x plane for a given value of $t = t_o$, inside which the correct curve $u(x, t_o)$ lies. That shaded region becomes thinner as t_o increases. Its width is negligible compared to unity. A significant quantity, which measures the steepness of the wave front is defined by

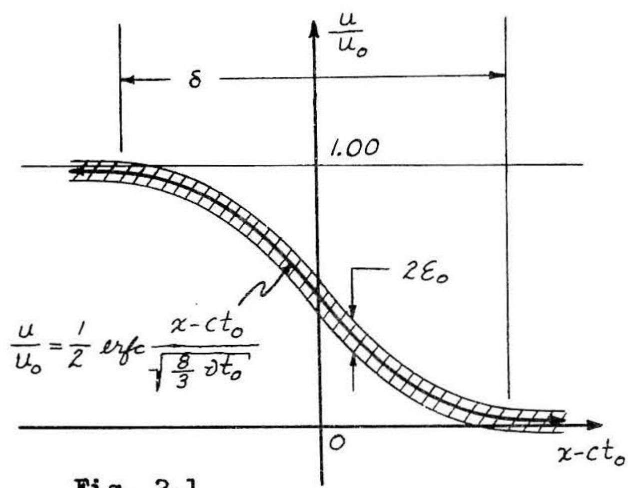

Fig. 2.1

$$\delta = 2\int_{ct_o}^{\infty} \frac{u}{u_o}\,dx \tag{2.214}$$

In the table below, the remainder term $\mathcal{E}_o$, the quantity δ and the distance $L = ct_o$ traveled by the center of the disturbance are calculated for various values of the time interval t_o. The process takes place in air under standard conditions, so that $\nu = 0.15$ cm²/sec and $c = 33{,}100$ cm/sec :

t_0	$\mathcal{E}_0$	$\delta(t)$	L
1 sec	6.45×10^{-6}	2.10 cm	33,100 cm
2 sec	4.55×10^{-6}	3.03 cm	66,200 cm
5 sec	2.87×10^{-6}	4.81 cm	165,500 cm
10 sec	2.05×10^{-6}	6.76 cm	331,000 cm
20 sec	1.45×10^{-6}	9.57 cm	662,000 cm

(2.215)

Since at the center of the wave $\frac{u}{u_0} = \frac{1}{2}$, it is clear from Table (2.215) that the asymptotic formula (2.28) is quite accurate for values of t_0 of one second and larger. The Table is also of interest as it shows how the original sharp signal is diffused by the action of viscosity.

Most of the remarks made above in connection with the step signal also apply to the pulse or Dirac signal, whose propagation is described by the asymptotic formula (2.212). The main difference between the step signal and the pulse lies in the fact that the amplitude of the pulse signal decreases with the passage of time. This effect cannot be attributed to viscous dissipation, since no dissipation terms enter the equations; it is another consequence of the dispersive action of viscosity. This is seen if one thinks of a pulse as a positive step signal of large amplitude A followed after a short time interval Δt by an equally strong negative step-signal. When the signals are both sharp, there is a region of length $c\Delta t$ in which the signal has the strength A. But, as the step signals flatten out, the region of strength A has the length $c\Delta t - \delta(t)$; eventually δ becomes equal to $c\Delta t$; after that instant, there is no region of amplitude A. The preceding discussion merely points up the fact that the pulse signal is obtained as the time derivative of the step signal.

For small values of the parameter $\frac{c^2 t}{4/3\,\nu}$, the preceding descriptive is not valid. Formula (2.29) applies to this case. It shows that initially the disturbance spreads like heat. The quantity $\frac{x}{\sqrt{t}}$ is a natural coordinate. If the remainder term ε is neglected, formula (2.29a) is a solution of the equation:

$$u_{tt} = \frac{4}{3}\nu u_{xxt} \tag{2.216}$$

There is an initial balance between viscous forces and acceleration. Only later, as the disturbance develops and pressure forces become important, does it approach the shape of a signal with a steep front.

Non-linear case: There is some hope that the corresponding non-linear problem may be studied. It is known that the non-linear inertia terms have a tendency to steepen a compression wave and this is counteracted by the flattening dispersive effect. When these two tendencies balance each other, a shock wave is obtained. The fully-developed shock wave has been studied taking viscosity and heat conduction into effect. In this case one introduces a system of coordinates moving with the shock. For the stationary shock the time derivatives then vanish and the equation corresponding to (2.24a) takes the form of a non-linear ordinary differential equation. An approximate method of treating this shock and showing the approach of a non-steady solution to a shock wave is given in Appendix B.

§2.3 One-Dimensional Transversal Waves.

In the piston problem studied above, a plane (the piston) was supposed to move perpendicular to its face. If instead it is moved parallel to itself a transversal wave is generated. In this case the boundary condition at the plane must be of a different type. It is not sufficient to have viscosity in the equation alone. In order to produce a wave at all by a plane moving parallel to itself one needs a no-slip condition, or some modification thereof. This result is very familiar from incompressible fluids. If we represent the plane by the entire x-axis, leaving out the z coordinate altogether, and consider the flow in the upper half of the (x,y) plane, all the quantities involved will be functions of y and t only. For our problem the flow will always be considered at rest initially and at $y = \infty$.

If now the velocity vector is given by $\vec{Q} = \varepsilon \vec{q} + \varepsilon^2 \vec{q}_2 + \ldots$, where $\vec{q} = u\vec{i} + v\vec{j}$, the boundary conditions at $y = 0$ are:

$$\text{no slip} \qquad u(0,t) = g'(t) \tag{2.31a}$$

$$v(0,t) = 0 \tag{2.31b}$$

Here, $x = x_0 + \varepsilon\, g(t)$ describes the motion of a point originally at $x = x_0$. The other boundary condition insures damping of the motion as $y \longrightarrow \infty$

$$u(\infty,t) = 0 \tag{2.32a}$$

$$v(\infty,t) = 0 \tag{2.32b}$$

The linearized system (1.37) is then simplified to:

$$u_t = \nu u_{yy} \tag{2.33a}$$

$$v_t + c^2 s_y = \frac{4}{3} \nu v_{yy} \tag{2.33b}$$

$$s_t + v_y = 0 \tag{2.33c}$$

with the initial conditions

$$u = v = s = 0 \qquad \text{at} \qquad t = 0 \tag{2.34a}$$

and the additional boundary conditions:

$$u(0,t) = g'(t) \tag{2.34b}$$

$$v(0,t) = 0 \tag{2.34c}$$

$$v(\infty,t) = 0\,, \quad u(\infty,t) = 0 \tag{2.34d}$$

Here u obeys the equation for a transversal wave. There is no coupling between u and v; the transversal waves generate no pressure waves. It will be seen later that this is no longer true when non-linear terms are taken into account, or when there is also a dependence on x (as for a half-infinite plate). The equations for v are satisfied by assuming v and s to be identically zero. Thus the problem reduces to that of solving the one-dimensional heat equation for u.

In particular, if the plate is started impulsively from rest and then maintained at constant velocity u_0, $g'(t)$ is a step function. The solution is then

$$\begin{aligned} u &= u_0 \operatorname{erfc} \frac{y}{2\sqrt{\nu t}} & \qquad t > 0 \\ u &= 0 & \qquad t < 0 \end{aligned} \tag{2.35}$$

The function erfc is discussed in §3.3 below and in Ref. 12, Appendix II. If, instead, the plate is started suddenly with an infinite velocity and then immediately stopped, the velocity prescribed at the boundary is a Dirac function (2.210b). The solution is then

$$u = \frac{\ell_0 y}{2\sqrt{\nu t^3}} \; e^{-\frac{y^2}{4\nu t}} \tag{2.36}$$

where ℓ_0 is a constant having the dimension length.

If a radiation problem is considered instead of a boundary-value problem, the force exerted by the plate is prescribed, rather than the velocity of the plate. If this force is a Dirac function, the familiar heat-source solution results (cf. Ref. 12, p. 219):

$$u = \sqrt{\frac{\nu}{t}}\; e^{\frac{-y^2}{4\nu t}} \tag{2.37}$$

Note that (2.36) is obtained from (2.37) by taking the y derivative and multiplying by $-\ell_o$.

There is a striking difference between such a transversal wave and the longitudinal waves discussed previously. The compressibility, as represented by the isentropic speed of sound, c, does not enter the equation, so the characteristics $x \pm ct$ = constant cannot play any role. However, this does not contradict the previous result that the propagation of initially sharp disturbances is determined by the non-viscous characteristics together with a dispersive effect. In this case a different set of characteristics has to be used. In a non-viscous fluid, discontinuities in pressure and normal velocity are propagated only along the ordinary hyperbolic characteristics whereas discontinuities in tangential velocity are propagated along the streamlines. Actually in a non-viscous, non-conducting fluid the streamlines are also characteristics (except in the case of potential flow, where irrotationality is postulated in the equations). This fact, sometimes overlooked, is stated explicitly in Ref. 1, Vol. II, p. 375, for example. It applies to both stationary and non-stationary flow and is the basis for the classical laws about propagation of vorticity. In the present case the streamlines are the lines y = constant in the (y, t)-plane (the displacements due to perturbation velocities are neglected in the linearized theory). Thus discontinuities may exist across these lines. This checks with the fact that for $v=0$, (2.33a) reduces to: $u_t=0$ or $u=f(y)$. f arbitrary. Thus, if at $t=0$, $u=u_0$ along the plate $(y=0)$ and $u=0$ elsewhere the discontinuity does not spread into the fluid $(y>0)$ but remains unchanged for $t>0$.

Remarks on Boundary Conditions. It can now be seen that, from a more general point of view, the viscous longitudinal waves and the viscous transversal waves are not very different in their relationship to the corresponding non-viscous solutions. Actually the important condition that has to be given up in passing from a viscous case to a non-viscous case is the requirement that the solution should be continuous. This requirement may be regarded as a generalized boundary condition. In the non-viscous case the discontinuities in the solution can occur across the characteristics. These discontinuities are replaced in the viscous case by a rapid but continuous transition which might be called a boundary layer. This state of affairs exists for both longitudinal and transversal waves.

For example, in non-viscous flow past a flat plate at zero angle of attack a condition on the tangential velocity at the plate may be imposed if discontinuities are allowed. In this case the discontinuity exists across the streamline along the plate and is a jump from the boundary value to the value normally assumed in the non-viscous flow. In the viscous case this discontinuity is replaced by a rapid transition layer which is the ordinary boundary layer. Of course the same argument applies to any slipstream in a non-viscous fluid. The similar relationship of the ordinary sound wave to the viscous longitudinal wave has been mentioned before.

In both longitudinal and transverse cases simplified methods for finding the solution in regions of rapid transition are sometimes used. These methods are exemplified by the classical boundary-layer theory and a similar procedure for longitudinal waves, in which the equations of motion are simplified by an assumption of rapid changes in a narrow layer. However, in the actual case such a description is adequate only for certain regions. In particular, it wasshown before that for the longitudinal wave the description above does not fit the region of small values of x and t. This corresponds to the failure of the classical boundary-layer theory near the leading edge of a flat plate. Furthermore in both cases the transition region thickens for large time (t) or far downstream (if the stationary case is considered). Thus the term "rapid" requires a comparison of some kind. Finally, it should be remembered that the one-dimensional cases are the simplest and that some complications may be introduced when higher-dimensional waves are considered.

Non-Linear Case. An iteration procedure for including some non-linear terms is discussed in Appendix C. The transversal wave in u is coupled with a longitudinal wave through the non-linear dissipation terms. The heat generated by the shear (as expressed by the energy equation) causes an expansion and thus a longitudinal pressure wave.

§2.4 General Remarks on Higher-Dimensional Waves

In §1.5, the notion of longitudinal and transversal wave is defined by means of equations (1.51) and (1.53). Some one-dimensional examples are given in the two preceding sections. In the present section (and in some of the following sections), we try to get a more intuitive picture of the two types of waves in higher dimensions. This is comparatively easy in the case of longitudinal waves because of their similarity to non-viscous waves (cf. equation 2.25). By analogy with the non-viscous theory and with the one-dimensional viscous longitudinal waves, one is led to consider viscous waves generated by a "spherical piston". If a wave is generated by a sphere which expands and contracts according to some "piston curve", it follows from symmetry considerations that there are no tangential viscous forces, and that the wave generated is irrotational. Furthermore, all the forces are either pressure forces or normal longitudinal viscous stresses. Thus, we have a longitudinal wave described by (1.51), with $\Delta \varphi = \varphi_{rr} + \frac{2}{r} \varphi_r$ because of spherical symmetry, and the external force $\vec{X} = 0$. The problem of the spherical piston is solved by separating variables (Ref. 53). Especially simple solutions are obtained by letting the radius of the pulsating sphere approach zero and specializing the signal to a step function or a pulse (Dirac function). The technique of fundamental solutions (Appendix D) can also be used. More general longitudinal waves are then obtained by superposition of basic solutions. In particular, two-dimensional waves are constructed by a superposition procedure which makes the waves independent of one coordinate. This is the method of descent which involves integration of basic solutions along a certain line. A similar integration procedure gives stationary solutions (see §2.6).

In the linearized theory of a non-viscous fluid, the motion of a symmetrical thin wing at zero angle of attack may be thought of as that of a symmetrical finite piston of variable position, which sends out pulses in both directions perpendicular to the plane of the wing. The same is true for longitudinal waves in a viscous fluid, if no conditions are imposed on the tangential velocity at the surface of the wing.

We also try to generalize the transversal wave due to a sliding infinite plate (§2.3) to higher-dimensional spaces. This is done in two different ways.

First, we study waves produced by a rotating circular cylinder. Such a cylinder resembles a sliding infinite flat plate, in that its boundary is transformed into itself by the motion, so that only transversal waves are generated. Expressions for such waves are obtained by solving simple boundary-value problems. Limiting cases of these are constructed by giving either the stream function or the vorticity some prescribed (symmetric) singularity at the origin. Another solution of particular importance is the vorticity dipole; its vorticity has the singularity of a two-dimensional heat dipole (instantaneous, or stationary in a moving fluid.)

Another way of generalizing the flow along the sliding infinite plate is to assume that the plate extends over only part of the x-axis. Under these conditions, longitudinal waves are generated along with the transversal waves. An important example of this type of flow is the flow past an infinitely thin semi-infinite flat plate at zero angle of attack (cf. Introduction). This problem is discussed (although not solved) in Chapter 3. By analysing the boundary conditions (§3.1), one finds that the semi-infinite flat plate may be considered as a superposition of similar finite plates. In the limit, one may allow the length of these component plates to tend to zero, while their retarding action (i.e. the value of u prescribed at the plate) tends to infinity. Such a plate is called a <u>singular flat plate</u>. Unless otherwise specified, that plate is located at the origin of the system of coordinates, which is the singular point in the flow field. In the non-stationary case, the action of the plate is assumed singular in time also: the plate acts on the fluid during an infinitesimal time interval, and is then removed from the fluid. It is not obvious a priori whether a singularity persists in the flow field after the plate has been removed. A full discussion of the boundary conditions for this problem is given in §3.1. At present, it is sufficient to point out that except at the singular point the solution for a singular flat plate must satisfy (1.37) or (1.43) with external force $\vec{X}=0$, in a two-dimensional field. The flow must be symmetric with respect to the x-axis, so that u is an even function of y, and v, u_y, ω are odd functions of y. Except at the origin, the velocity components and their derivatives must be continuous. On the x-axis, including the origin, v should be zero. Thus there are the following conditions for the stationary case:

For all x, including $x = 0$

$$v(x,0) = 0 \tag{2.41a}$$

$$v(x,\infty) = 0 \tag{2.41b}$$

and, except for $x = 0$

$$u_y(x,0) = 0 \tag{2.42}$$

which, because of (2.41a), is equivalent to

$$\omega(x,0) = 0 \tag{2.42'}$$

In the non-stationary case the same requirements should be fulfilled for all values of t. To solve such a degenerate boundary-value problem is equivalent to finding the effect of a singular shearing stress at the origin. Formulated in mathematically rigorous terms, the problem consists in finding the fundamental solution (cf. §2.1) of the systems (1.37) and (1.43). This is done in Appendix D, where the problem is reduced, by means of a Laplace transformation, to finding the fundamental solutions of some simple second-order equations. The results of this analysis are presented and discussed in §2.8. The solution is given in the form of a contour integral which requires considerable further analysis.

However, before presenting the fundamental solution, we attempt to solve the singular flat plate problem by two other methods. These are less rigorous than the mathematical technique of Appendix D, and lead only to incomplete results; but they are much more intuitive, and it is hoped that they contribute to the understanding of one of the fundamental problems examined in this report: the induction of longitudinal waves by transversal waves, and the related question of how the boundary-layer solution of a problem compares with the solution of the full equations of motion. Similar methods are also used, later, for problems whose complete solution is not known.

First, the vorticity field of the singular flat plate is analysed. It proves to have dipole structure. One therefore expects to find the solution of the singular flat plate problem by forming the y derivative of the vorticity pole flow field. One obtains in this way a transversal wave which satisfies conditions (2.41) and (2.42), except that $v(0,0)\neq 0$. It seems natural to try the addition of a longitudinal wave which makes the resultant flow satisfy the condition $v(0,0)=0$ without disturbing the other conditions. Such a wave can be constructed by the methods discussed in §2.1. A comparison with the fundamental solution shows this intuitive construction to be correct.

The second method (§2.7) consists in constructing a solution of the transversal wave equation for the component u. When one attempts to complete the transversal wave by determining v from the continuity equation, one finds that the boundary conditions on v are not all satisfied. This is remedied by the addition of a longitudinal wave. But now, other boundary conditions are not satisfied, and this method leads to a somewhat questionable iteration procedure. It is thus inferior to the previous methods. But, as the next chapter will show, it is closely related to boundary-layer theory, and thus has particular interest. This fact is also brought out by comparing it with the solution of the boundary-layer equations.

Generally, three-dimensional transversal waves are not discussed since two-dimensional waves display the complication of the problem sufficiently. However, in Appendix D, the fundamental solutions for three-dimensional waves are derived.

§2.5. Higher-Dimensional Longitudinal Waves. Methods of Descent.

The discussion in this section is mainly from an intuitive point of view. Many results are similar to those found for one-dimensional flow (§2.2) so that the analysis is not carried out in detail.

Non-stationary waves with spherical symmetry: A simple generalization of the results found for the one-dimensional piston problem (§2.2) is obtained from the study of waves generated by a pulsating sphere. It follows from symmetry considerations that the waves are longitudinal and may be described with the aid of a velocity potential φ which satisfies (cf. 1.52)

$$\varphi_{tt} - c^2 \Delta \varphi = \frac{4}{3} \nu \Delta \varphi_t \tag{2.51a}$$

$$\Delta \varphi = \varphi_{rr} + \frac{2}{r} \varphi_r \; ; \quad r^2 = x^2 + y^2 + z^2 \tag{2.51b}$$

The boundary-value problems associated with the pulsating sphere are treated by standard methods. In Ref. 53, the variables are separated. One obtains especially simple solutions by letting the radius of the sphere approach zero.

The solutions of radiation problems with spherical symmetry also give simple longitudinal waves. For example, the velocity potential of the fundamental solution of system (1.52) is, within a constant factor,

$$\varphi(r,t) = \frac{1}{2\pi i r} \int_{-i\infty}^{i\infty} \frac{e^{\left(\sigma t - \frac{\sigma r}{\sqrt{c^2 + \frac{4}{3}\nu\sigma}}\right)}}{c^2 + \frac{4}{3}\nu\sigma} \, d\sigma \tag{2.52}$$

This formula is derived in non-dimensional form in Appendix D (Part c).

It should be noted that the function defined by the Laplace integral is related to the functions previously discussed in the one-dimensional case (See 2.26 and 2.211) by simple differentiations. Asymptotic evaluations of (2.52) may be carried out by the methods used in §2.2. When the local Reynolds numbers $\frac{c^2 t}{4/3\,\nu}$ and $\frac{c r}{4/3\,\nu}$ (cf. 2.27) are large, the relationship between viscous and non-viscous waves is the same as in the one-dimensional case. For this reason, further mathematical analysis is not presented here.

Non-Stationary Waves with Cylindrical Symmetry. Cylindrical longitudinal waves are treated in the same general manner as spherical waves; the pulsating sphere is replaced by a pulsating cylinder, and the operator Δ is now $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$. Some examples of these waves are given in Ref. 53. In particular, the fundamental solution is (cf. Appendix D):

$$\varphi(r,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{\sigma t'}}{1+\sigma} K_0\left(\frac{\sigma r'}{\sqrt{1+\sigma}}\right) d\sigma \tag{2.53a}$$

$$\text{where} \quad r^2 = x^2 + y^2, \quad t' = \frac{c^2 t}{4/3\,\nu}, \quad r' = \frac{c r}{4/3\,\nu} \tag{2.53b}$$

Since formula (2.52) is easier to analyse than (2.53), it is sometimes convenient to construct cylindrical waves from spherical waves by a method of descent. For instance, the cylindrical pulse is considered as a uniform distribution of spherical pulses along the z-axis. Formula (2.53) then follows from (2.52) if one uses the identity:

$$\int_{-\infty}^{\infty} \frac{e^{-A\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\, dz = 2K_0\left(A\sqrt{x^2+y^2}\right) \tag{2.54}$$

<u>General Non-Stationary Longitudinal Waves:</u> A symmetrical wing at zero angle of attack which moves with arbitrary velocity and acceleration in the plane $y=0$ in an inviscid fluid behaves as a finite piston. At each instant, a vertical velocity is imparted to both sides of the plane $y=0$. The vertical velocity vanishes off the wing, and is proportional to the local slope and instantaneous forward velocity of the wing. This problem resembles the piston problem discussed in §2.2, the only difference being that the function $f(t)$ (the piston curve) now depends on x and z also. In the non-viscous case, the problem is solved by assuming a distribution of spherical pulses in the plane $y=0$, the pulse strength being proportional to the local instantaneous vertical velocity.

To formulate an equivalent boundary-value problem for a viscous fluid, one assumes no conditions on the tangential (horizontal) velocity component, but only on the normal (vertical) component. The solutions of such problems are useful later in superposition procedures. Then the qualitative ideas described above may be carried over from the non-viscous to the viscous fluid. Thus, a moving symmetrical wing creates longitudinal viscous waves, generated in effect by a distribution of spherical pulses in the plane $y=0$. However, the relation between source strength and prescribed vertical velocity on the wing must be reconsidered. This problem is not discussed in this report. One should expect the correction for viscosity to be qualitatively unimportant.

<u>Stationary Waves. Second Method of Descent:</u> The discussion is confined to two-dimensional waves. The equation for the velocity potential is

then (cf. 1.56e)

$$(M^2-1)\varphi_{xx} - \varphi_{yy} = \frac{4}{3}\,\frac{\nu U}{c^2}\,(\varphi_{xxx} + \varphi_{xyy})$$

The third-order terms determine the characteristics of the equation. Two sets of characteristics are imaginary, indicating the partly elliptic nature of the equation. In addition, the lines y = constant are characteristics. These might be called parabolic, but they do not represent double roots of the characteristic equations, as the t = constant lines did in the theory of non-stationary waves. Thus, even in the case of supersonic flow, disturbances spread over the entire plane (including the upstream region). This becomes clear if the stationary case is regarded as a limit of the non-stationary case (cf. §1.4). The propagation of the disturbance upstream is due to the infinite signal velocity of the corresponding non-stationary waves. Again, the characteristics do not depend on compressibility; but the Mach number M determines the subcharacteristics (cf. Introduction p.10). In a non-viscous fluid, one-dimensional non-stationary and two-dimensional stationary supersonic waves are described by the same equation. This is no longer true for a viscous fluid as a comparison of (1.52) and (1.56) shows. The analogy is spoiled by the presence of the term φ_{xxx} in (1.56). Removal of this term would make x = constant a double set of parabolic characteristics equivalent to t = constant in (1.52), and destroy the elliptic character of (1.56).

The fundamental solution of (1.56e) may be obtained directly by the methods of Appendix D. It (and other solutions) may also be obtained by a limiting procedure based on the ideas of §1.4. That procedure is actually a method of descent. Assume that the center of a cylindrical pulse travels along the x-axis in the negative direction with velocity U and passes through the origin at $t=0$. By this we mean that at each instant $t=\tau$ a cylindrical pulse is emitted at the point $\xi=-U\tau$, $\eta=0$. If this process has been going on since $\tau=-\infty$ and if $F(x-\xi, y-\eta, t-\tau)$ is the fundamental solution of the two-dimensional non-stationary problem then the effect at (x,y) of the travelling cylindrical pulse is

$$F_5 = \int_{-\infty}^{t} F(x+U\tau, y, t-\tau)\, d\tau \tag{2.55}$$

Introduce the Galilean transformation

$$\begin{aligned} \bar{x} &= x + Ut \\ \bar{y} &= y \\ \bar{t} &= t \end{aligned} \tag{2.56}$$

In the new system of coordinates the pulse is at rest at the origin. With (2.56) and the change of integration variable $T=\bar{t}-\tau$, F_5 becomes

$$F_5(\bar{x}, \bar{y}) = \int_{0}^{\infty} F(\bar{x}-UT, \bar{y}, T)\, dT \tag{2.55'}$$

In the new system of coordinates the wave is thus seen to be stationary.

Examples of this process will be given in subsequent sections. (See 2.613, 2.614, 2.77). It has not been carried out analytically for the present case. Neither has the fundamental solution been determined directly. This would be a straightforward procedure by the methods of Appendix D. However, the fundamental solution for the full equations will have a longitudinal part which is sufficient for our purposes. It is discussed in §2.8.

§2.6. Two-dimensional Transversal Waves. Vorticity Waves.

Some simple boundary-value problems for non-stationary transversal waves are discussed first. Degenerate cases of these lead to especially simple solutions with singularities. The stationary case is then treated, both by considering it as a limiting case of non-stationary flow and directly by constructing solutions with the appropriate singularities. Three-dimensional flow is not taken up in this section, but the solutions can be deduced from those in Appendix D.

Non-stationary waves:

Rotating cylinder: Consider an infinite circular cylinder of radius r_o with center at the origin, in a viscous fluid at rest before $t=0$. Let it start rotating at $t=0$ with angular velocity $f(t)$. The resulting flow pattern consists of a transversal wave only, since there is no motion of the solid boundary in a direction normal to the boundary surface. The solution is therefore independent of compressibility effects. This classical boundary-value problem is discussed in Ref. 18, p. 212 (where many additional references are quoted). Only the results need be given here.

It is found that a stream function ψ may be constructed, which satisfies the heat-conduction equation and vanishes at infinity. The equations of motion are then:

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_r \qquad \text{(radial velocity component)} \qquad (2.61a)$$

$$-\frac{\partial\psi}{\partial r} = u_\theta \qquad \text{(tangential velocity component)} \qquad (2.61b)$$

$$\psi_t = \nu\Delta\psi \qquad (2.61c)$$

It follows from (2.61) that

$$\Delta \psi = \omega \tag{2.61d}$$

The boundary conditions for the rotating cylinder are

$$u_\theta(r_o, \theta) = -\frac{\partial \psi}{\partial r}(r_o, \theta) = r_o f(t) \tag{2.62a}$$

$$\psi(\infty, \theta) = 0 \tag{2.62b}$$

Note that ψ satisfies the same type of boundary conditions as the potential of longitudinal waves produced by a pulsating cylinder.

Since boundary conditions (2.62) have rotational symmetry, the solution of (2.61c) enjoys the same property. Therefore:

$$u_r = 0 \tag{2.63a}$$

$$\frac{\partial u_\theta}{\partial \theta} = 0 \tag{2.63b}$$

from which it follows that

$$\omega = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \tag{2.63c}$$

$$\frac{\partial \omega}{\partial \theta} = 0 \tag{2.63d}$$

The simple case in which $f(t)$ is a unit step function is given as an example. It yields

$$u_\theta = \frac{u_o}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_1\left(r\sqrt{\frac{s}{\nu}}\right)}{K_1\left(r_o\sqrt{\frac{s}{\nu}}\right)} e^{st} \frac{ds}{s} \tag{2.64a}$$

with the vorticity distribution

$$\omega = \frac{u_o}{4\pi i r} \int_{-i\infty}^{i\infty} \frac{\left[K_0\left(r\sqrt{\frac{s}{\nu}}\right) + K_2\left(r\sqrt{\frac{s}{\nu}}\right)\right] r\sqrt{\frac{s}{\nu}} + 2K_1\left(r\sqrt{\frac{s}{\nu}}\right)}{K_1\left(r_o\sqrt{\frac{s}{\nu}}\right)} e^{st} \frac{ds}{s} \tag{2.64b}$$

These should be compared with the corresponding formulas for the flat plate (2.35).

Since the flow field is easily determined from the stream function by use of (2.61a,b), one might expect to find simple flow fields by giving ψ a simple singularity. Thus, if ψ has the form of a heat source, the following field results:

$$\psi = \frac{1}{t} e^{-\frac{r^2}{4\nu t}} \tag{2.65a}$$

$$u_\theta = \frac{r}{2\nu t^2} e^{-\frac{r^2}{4\nu t}} \tag{2.65b}$$

$$\omega = \left(1 - \frac{r^2}{4\nu t}\right) \frac{e^{-\frac{r^2}{4\nu t}}}{\nu t^2} \tag{2.65c}$$

However, it turns out to be more fruitful to study waves for which the vorticity, rather than the stream function, has simple singularities. In particular, we investigate the case when ω has a simple pole at the origin, and the case when it has a dipole. The latter is directly related to the flat plate problem.

Vorticity pole: The vorticity pole may be thought of as the limiting solution of the following boundary-value problem: A cylinder of very small radius is rotated very rapidly during a very short time interval, and then the fluid is allowed to slip. The solution is then (cf. 2.65):

$$\omega = \frac{1}{t} e^{-\frac{r^2}{4\nu t}} \tag{2.66}$$

Note that in the limit $(r_0 \to 0)$, u_θ at the cylinder does not become a Dirac function. After the initial pulse, u_θ is not prescribed to vanish, but left undetermined. An alternate interpretation makes the problem a radiation problem where the force, rather than the boundary value, is prescribed. The vorticity pole is generated when the rotation of a solenoidal force field (curl $\vec{X}$ in equation 1.311a) is a Dirac function in space and time. This interpretation explains why ω rather than ψ behaves like the disturbance due to an instantaneous heat source (cf. also §2.3 and especially equation 2.37).

Generally, when equations (2.61) hold, one determines the flow field from the vorticity field with the aid of the stream function, by solving the Poisson equation (2.61d). But for a problem with rotational symmetry (described by 2.63) there is only an ordinary differential equation to solve

$$\frac{du_\theta}{dr} + \frac{u_\theta}{r} = \omega(r) \tag{2.67a}$$

with boundary condition

$$\lim_{r \to \infty} r u_\theta = 0 \tag{2.67b}$$

When $\omega(r)$ takes the value given by (2.66), equation (2.67) becomes:

$$\frac{d}{dr}(r u_\theta) = \frac{r}{t} e^{-\frac{r^2}{4\nu t}}$$

or

$$r u_\theta = -2\nu e^{-\frac{r^2}{4\nu t}} + g(t)$$

where $g(t) \equiv 0$ because of (2.67b).

The flow field associated with a vorticity pole is thus:

$$u_\theta = \frac{-2\nu}{r} e^{-\frac{r^2}{4\nu t}} \tag{2.68a}$$

$$u_r = 0 \tag{2.68b}$$

or

$$u = \frac{2\nu y}{r^2} e^{-\frac{r^2}{4\nu t}} \tag{2.68c}$$

$$v = \frac{-2\nu x}{r^2} e^{-\frac{r^2}{4\nu t}} \tag{2.68d}$$

It follows from the last two equations that the flow field is conveniently described in terms of the stream function ψ

$$\psi \equiv \psi_0(r) = -2\nu \int_r^\infty \frac{e^{-\frac{\xi^2}{4\nu t}}}{\xi} d\xi \tag{2.69a}$$

$$u = \psi_y \qquad v = -\psi_x \tag{2.69b}$$

Note that, as $t \to \infty$, the flow is not damped out, but approaches the limiting flow:

$$u = \frac{2\nu y}{r^2} \tag{2.610a}$$

$$v = -\frac{2\nu x}{r^2} \tag{2.610b}$$

This flow may be obtained from a stream function

$$\psi = 2\nu \log r \tag{2.610c}$$

Here ψ is normalized so that the streamline $\psi = 0$ is at $r = 1$ rather than at $r = \infty$ as in (2.69a). This is the familiar vortex flow of an incompressible inviscid fluid. Its vorticity vanishes everywhere except at the origin. Since it is given by a harmonic function independent of time, it satisfies the equations for both longitudinal and transversal waves. Within the linearized theory, the vortex flow is thus a permanent configuration for a viscous fluid.

It may be argued that the solution (2.610) should be subtracted from (2.68), to obtain damping as $t \to \infty$. However, there seems to be no compelling reason for requiring such damping. Furthermore, if (2.610) is subtracted, u and v do not vanish at t = 0. This would imply more than the infinite signal velocity characteristic of parabolic equations. It would indicate an instantaneous adjustment of $\vec{q}\,(r)$ to a finite value under the action of a force applied a finite distance r away. Such an effect is possible only for an incompressible fluid, in which pressure waves spread instantaneously throughout the field; this is expressed by the elliptic character of the equations of motion. Actually, the dipole counterpart of (2.610) will appear in the longitudinal component of the wave due to a singular flat plate (cf. §2.8).

Vorticity dipole: It is noted that differentiation of (2.66) with respect to y gives an expression which has the symmetry properties required of the vorticity field associated with the singular flat plate. The singularity at the origin has the nature of a dipole.

$$\text{Vorticity Dipole:} \quad \omega = \frac{-y}{2\nu t^2} e^{-\frac{r^2}{4\nu t}} \tag{2.611}$$

If one tries to obtain the dipole flow field by solving the Poisson equation (2.61d), a complicated boundary-value problem results. However, the u and v components of the transversal part of the flow are found by differentiating (2.68c, d) with respect to y. The field so obtained obviously has the vorticity (2.611), and u is symmetric with respect to the x-axis while v is anti-symmetric. The only singularity is at the origin, so that $v(x,0)=0$ for $x \neq 0$. At the origin, there is a singularity in both u and v. All the conditions for the singular flat plate are thus satisfied except $v(0,0)=0$. However, a longitudinal wave of the type discussed in § 2.5 may be added, which has a singularity in v at the origin, to cancel the singularity of the transversal wave, without disturbing the other boundary conditions. Analytical details will be given in § 2.8 and Appendix D. The transversal component of the wave generated by the singular flat plate is therefore:

$$u = \frac{\partial^2 \psi_0}{\partial y^2} \tag{2.612a}$$

$$v = -\frac{\partial^2 \psi_0}{\partial x \partial y} \tag{2.612b}$$

where ψ_0 is defined by (2.69a).

Two-dimensional stationary waves.

Vorticity pole and Dipole: The interpretation of vorticity pole and dipole flows as limiting cases of non-stationary boundary-value problems leads to some difficulty; it is more straightforward to consider them as limiting cases of radiation problems. Indeed, the flows can be constructed by a superposition of the solutions (2.68, 2.612) (method of descent. See Eq. 2.55'). Assume that vorticity singularities of the proper type and of uniform strength are generated along the x-axis in such a manner that a single singularity of constant strength moves to the left along the x-axis with velocity U. By the Galilean transformation (1.41), a stationary flow field is then obtained.

Thus, in the case of a vorticity pole, one finds:

$$\omega \equiv \omega_o(x,y) = \int_0^\infty e^{-\frac{(x-Ut)^2+t^2}{4\nu t}} \frac{dt}{t} = e^{\frac{Ux}{2\nu}} \int_0^\infty e^{-\frac{rU}{4\nu}(\tau+\frac{1}{\tau})} \frac{d\tau}{\tau} = e^{\frac{Ux}{2\nu}} K_o\left(\frac{Ur}{2\nu}\right) \qquad (2.613)$$

and for the vorticity dipole,

$$\omega = \omega_1(x,y) = \int_0^\infty -\frac{y}{2\nu t^2} e^{-\frac{(x-Ut)^2+y^2}{4\nu t}} dt = \frac{Uy}{2\nu} e^{\frac{Ux}{2\nu}} \int_0^\infty e^{-\frac{rU}{4\nu}(\tau+\frac{1}{\tau})} \frac{d\tau}{\tau^2} = -\frac{Uye^{\frac{Ux}{2\nu}}}{2\nu r} K_1\left(\frac{Ur}{2\nu}\right) \qquad (2.614)$$

The same results can be obtained directly by constructing solutions of the equation for stationary vorticity propagation (1.45) with simple pole and dipole singularities at the origin. In the two-dimensional case, that equation is:

$$\omega_{xx} + \omega_{yy} - \frac{U}{\nu}\,\omega_x = 0 \qquad (2.615)$$

Here the singular solution cannot be expected to depend on $r=\sqrt{x^2+y^2}$ only, because the term ω_x destroys the symmetry of the equation about the origin. This is due to the fact that the x-direction is distinguished as the direction of the mean flow U. However, by the separation of an exponential factor, an equation with the required symmetry can be obtained. (Ref. 1, Vol. II, p. 137). Define a function Ω by

$$\Omega(x,y) = e^{-\frac{Ux}{2\nu}} \omega(x,y) \tag{2.616}$$

Then (2.615) is equivalent to

$$\Omega_{xx} + \Omega_{yy} - \frac{U^2}{4\nu^2} \Omega = 0 \tag{2.617}$$

The "pole" solution of this equation is

$$\Omega = K_0\left(\frac{Ur}{2\nu}\right) \tag{2.618}$$

where $r^2 = x^2 + y^2$

and the dipole is

$$\Omega = \frac{\partial}{\partial y}\left[K_0\left(\frac{Ur}{2\nu}\right)\right] = -\frac{U}{2\nu}\frac{y}{r}K_1\left(\frac{Ur}{2\nu}\right) \tag{2.619}$$

The corresponding values of ω are then obtained by multiplying the above expressions by $e^{\frac{Ux}{2\nu}}$. This yields the previously obtained formulas (2.613) and (2.614). In interpreting these two formulas and similar formulas which arise later in the analysis, the following asymptotic expressions for K_0 and K_1 are useful (see Ref. 13, p. 374):

$$\lim_{z\to 0} K_0(z) \sim -\left[\log\frac{z}{2}+\gamma\right]\left[1+\frac{z^2}{4}+\ldots\right]+\left[\frac{z^2}{4}+\ldots\right] \tag{2.620a}$$

$$\lim_{z\to\infty} K_0(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1-\frac{1}{8z}\ldots\right] \tag{2.620b}$$

$$\gamma = .577\ldots \tag{2.620c}$$

$$\lim_{z\to 0} K_1(z) \sim -\frac{1}{z} - \left[\log\frac{z}{2}+\gamma\right]\left[\frac{z}{2}+\frac{z^3}{16}+\ldots\right]+\frac{z}{4}\ldots \tag{2.621a}$$

$$\lim_{z\to\infty} K_1(z) = -\sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1+\frac{3}{8z}+\ldots\right] \tag{2.621b}$$

These formulas, substituted into (2.613) and (2.614) suggest that, in a sense, the situation is similar to that encountered for the longitudinal waves. Vorticity is propagated in two different ways. It is propagated along the subcharacteristics, which, in this case, are the streamlines; this type of propagation actually means that vorticity is not propagated at all with respect to the fluid. But in addition, there is a viscous dispersion analogous to heat diffusion. The effect of this type of propagation is that vorticity spreads throughout the entire plane. The form of the vorticity field can be seen better if polar coordinates, defined by $x = r\cos\theta$ and $y = r\sin\theta$, are used. Then from (2.614) the dipole field is

$$\omega_1(r,\theta) = -\frac{U}{2\nu}\sin\theta\, e^{\frac{r\cos\theta}{2\frac{\nu}{U}}} K_1\left(\frac{Ur}{2\nu}\right) \tag{2.622a}$$

and asymptotically for large r

$$\omega_1(r,\theta) \sim -\frac{U}{2\nu}\sqrt{\frac{\pi\nu}{Ur}}\, e^{-\frac{Ur}{2\nu}(1-\cos\theta)} \sin\theta \tag{2.622b}$$

This shows that the vorticity introduced at the origin spreads mostly downstream $(x>0)$ and is exponentially damped for $x<0$.

The transversal velocity field associated with the stationary vorticity dipole may be found by descent from equations (2.612) in the same way that (2.614) was derived from (2.611). It will satisfy all boundary conditions for the singular flat plate except for a singularity in v at the origin. This singularity may be removed by adding a longitudinal wave. The explicit analytic expressions are derived by different methods in Appendix D and discussed in detail in §2.8.

§2.7 Transversal Waves of Boundary-Layer Type. Iteration Procedure.

It was seen in the preceding section that a transversal wave alone cannot satisfy the boundary conditions for a singular flat plate (stationary or non-stationary). In order to make $V=0$ at the plate a longitudinal wave has to be added. Using terminology from boundary-layer theory one might say that the transversal wave changes the apparent shape of the flat plate from an object of zero thickness to something like a wedge. It is this apparent thickness that induces the longitudinal wave. In order to bring out this connection better a new method for treating the singular plate will be discussed in the present section. The method will lead to an iteration procedure which is not very suitable for obtaining the complete solution. However, it has the advantage of being intuitive and showing why a transversal wave is insufficient. It also contributes to the understanding of the role of boundary-layer theory. This theory actually corresponds to the first step in the iteration procedure or an approximation thereto. This connection will be further clarified in the next chapter where the same iteration method is applied to the more conventional boundary-value problem of the half-infinite stationary flat plate.

First an expression for the horizontal component u is found which satisfies the conditions for the singular flat plate and the equations for transversal waves. If the complete solution were a transversal wave, then v could be determined from u with the aid of the continuity equation for transversal waves. But in integrating this first-order equation, only one of the two boundary conditions on v may be satisfied. This shows that transversal waves alone are not sufficient to solve the problem.

Non-Stationary Two-Dimensional Case: The appropriate equations are obtained from (1.53) with the external force $\vec{X}=0$. In view of (1.53a), u must satisfy the two-dimensional heat equation; the continuity equation (1.53b) is that of an incompressible fluid. Thus

$$u_t = \nu\,(u_{xx}+u_{yy}) \tag{2.71a}$$

$$u_x + v_y = 0 \tag{2.71b}$$

Since u is symmetric about the x-axis, the appropriate singular solution of (2.71a) corresponds to the instantaneous heat source (cf. the vorticity pole discussed in §2.6). It is

$$u(x,y,t) = \frac{1}{t}\, e^{-\frac{r^2}{4\nu t}} \qquad (r^2 = x^2+y^2) \tag{2.72}$$

Having u, the next step would be to try to determine v by a direct integration. However, there is only one constant of integration at our disposal, whereas v must satisfy two boundary conditions: $v(x,0) = v(x,\infty) = 0$ (2.41). This difficulty is typical of boundary-layer theory where the condition on v at $y=0$ is usually satisfied but the condition on v at $y=\infty$ is given up. In the present case we shall instead satisfy the condition at $y=\infty$. Then for $y \geq 0$

$$v = \int_y^\infty u_x\,dy = \frac{x}{\sqrt{\nu}\,t^{3/2}}\, e^{-\frac{x^2}{4\nu t}} \int_{\frac{y}{2\sqrt{\nu t}}}^{\infty} e^{-\sigma^2}\,d\sigma = \frac{x\,e^{-\frac{x^2}{4\nu t}}}{\sqrt{\nu}\,t^{3/2}} \left[\operatorname{erfc} \frac{y}{2\sqrt{\nu t}}\right] \frac{2}{\sqrt{\pi}} \tag{2.73}$$

For negative values of y, v is determined by the condition of antisymmetry: $v(x,y,t) = -v(x,-y,t)$. For the error function, see (3.38) and Ref. 12, Appendix II.

On the x-axis we have

$$v(x,0\pm,t) = \mp \frac{2x}{\sqrt{\pi}\,t^{3/2}}\, e^{-\frac{x^2}{4\nu t}} \qquad \begin{array}{l} -\text{sign for } y=0+ \\ +\text{sign for } y=0- \end{array} \tag{2.74}$$

Hence we have a transversal wave (u, v) which satisfies all boundary conditions for the singular plate as described in §2.4 except that v is discontinuous and $\neq 0$ on the x-axis.

A natural procedure which restores the correct values of v on the x-axis is the addition of a longitudinal wave. This wave is determined by the condition that v takes values on the x-axis which are (-1) times those given by (2.74). If we neglect viscosity insofar as the longitudinal waves are concerned, the solution is very easy. It is obtained by placing two-dimensional non-stationary pulses

$$\frac{S(\xi,0,\tau)}{\sqrt{c^2(t-\tau)^2-(x-\xi)^2-y^2}} \tag{2.75a}$$

along the x-axis. The pulse strength $S(\xi,\tau)$ is given by the formula

$$S(\xi,\tau)=\frac{v(\xi,0,\tau)}{2\pi} \tag{2.75b}$$

When $x<0$ the superimposed longitudinal wave v is positive for $0+$ and negative for $0-$. Hence the waves sent out are all compression waves. For $x>0$, the situation is reversed. It should be noticed that u as given by (2.72) is always positive. Thus the singular flat plate is moved in the direction of the positive x-axis.

The previous study of viscous longitudinal waves (§2.5) indicates that the above description gives a qualitatively correct picture. If viscosity is taken into account, the formula for the two-dimensional longitudinal pulse (2.75a) and the formula for the source strength will have to be modified. According to §2.5 this will not change the situation qualitatively. After addition of the longitudinal wave, conditions on v are satisfied both at $y=0$ and at $y=\infty$. However, conditions on u have been spoiled by adding the longitudinal wave. In particular u_y is discontinuous on the x-axis. When instead a finite or semi-infinite plate is studied by the same method (§3.3), it is seen that the conditions on u are again not satisfied. Thecretically, this error may be corrected by adding more transversal waves, etc. Actually, this is not a tractable method. Our interest is only in the first two steps. The first transversal wave is related to the solution given by boundary-layer theory. The added longitudinal wave represents (if viscosity is neglected) the correction of the outer potential flow for the apparent change in body shape due to the boundary layer. Only a comparison with the correct complete solution (see §2.8) can tell us about the validity of the iteration procedure. The connection with boundary-layer theory is seen more clearly in the discussion of the stationary case immediately below, and especially in §3.3, where the semi-infinite flat plate is discussed.

Two-Dimensional Stationary Case: This case may be treated either directly or as a limit of the non-stationary case (method of descent; see §2.5). The equations are (cf. 1.57)

$$U u_x = \nu\,(u_{xx} + u_{yy}) \tag{2.76a}$$

$$u_x + v_y = 0 \tag{2.76b}$$

The appropriate singular solution for u is of the source type (cf. 2.72 and 2.55'), namely:

$$u(x,y) = \int_0^\infty \frac{1}{t}\, e^{-\frac{(x-Ut)^2+y^2}{4\nu t}}\, dt = e^{\frac{Ux}{2\nu}}\, K_0\!\left(\frac{U\sqrt{x^2+y^2}}{2\nu}\right) \tag{2.77}$$

The associated v velocity which satisfies the boundary condition at $y=\infty$, is, for $y \geq 0$

$$v(x,y) = \frac{U}{2\nu} e^{\frac{Ux}{2\nu}} \left\{ \int_y^\infty K_0\left(\frac{U\sqrt{x^2+\eta^2}}{2\nu}\right) d\eta - x \int_y^\infty \frac{K_1\left(\frac{U\sqrt{x^2+\eta^2}}{2\nu}\right)}{\sqrt{x^2+\eta^2}} d\eta \right\} \quad (2.78)$$

In interpreting these formulas for u and v, the estimates of the K_0 and K_1 functions given by (2.620) and (2.621) may be used.

The compensating longitudinal wave is that generated by a symmetrical body whose surface has the slope $\frac{v'(x,0)}{U}$. This is the "bulging" of the body due to the boundary layer. Note that this "apparent body" now extends along the whole x-axis, although its thickness decreases exponentially upstream of the leading edge $(x<0)$. The longitudinal wave is obtained by solving (1.56) with the boundary condition $\Phi_y(x,0) = v'(x,0)$ equal to the negative of $v(x,0)$ given by formula (2.78), and with no condition imposed on u. In the supersonic case, pressure disturbances extend along the subcharacteristics into the outer fluid at the same time they are dispersed by viscosity (cf. §§2.8 and 3.4).

The added longitudinal wave spoils the boundary conditions on u and u_y, just as in the non-stationary case, and the iteration procedure must be continued.

Boundary-Layer Equations: If, instead of the complete transversal wave equations for u (2.76), the simplified boundary-layer equations (1.59) are considered, the situation is very similar. The solution corresponding to a singular flat plate at the origin is:

$$u = \frac{1}{\sqrt{x}} e^{-\frac{Uy^2}{4\nu x}} \qquad \text{for} \qquad x>0 \qquad (2.79a)$$

$$u = 0 \qquad \text{for} \qquad x<0 \qquad (2.79b)$$

The main difference between this solution and (2.77) consists in the fact that (2.79) shows no effect upstream of the leading edge while (2.77) decreases exponentially (cf. 2.614). An attempt to satisfy the continuity equation and boundary conditions, using the solution (2.79) for u leads to the same difficulties as those encountered above.

§2.8. Waves due to a Singular Shearing Force

In this section an interpretation will be given of some of the results derived in Appendix D. The relationship of the fundamental tensor Γ to the flow field generated by a singular shearing force will be shown. Then the exact solution for the full equations as expressed by this fundamental tensor will be compared with the previous intuitive results (§2.6, §2.7) for treating the problem of the singular plate. Thus an evaluation of these methods can be made. The discussion will be based mostly on the case of two-dimensional stationary flow, and some remarks will be made about the non-stationary case.

Stationary Case

Basic Formulas for Two-dimensional Fundamental Tensor: In this section we use the following notation:

(x,y) = coordinates in the plane

$\vec{q}$ = velocity vector at a point = $\begin{pmatrix} u \\ v \end{pmatrix}$

$\vec{X}$ = external force per unit mass = $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$\Gamma(x-\xi, y-\eta)$ = fundamental solution tensor = $\begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}$

Then from the definition of Γ

$$\vec{q}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x-\xi, y-\eta)\, \vec{X}(\xi,\eta)\, d\xi d\eta \tag{2.81}$$

Now if $\vec{X}$ is an impulse (Dirac) function located at (x_1, y_1) , for example,

$$\vec{X} = \frac{\vec{Z}}{4\varepsilon^2} \qquad \begin{array}{l} |x-x_1|<\varepsilon \\ |y-y_1|<\varepsilon \end{array} \tag{2.82}$$

$$= 0 \qquad \text{elsewhere,}$$

the integrals in (2.81) disappear as $\varepsilon \to 0$ and

$$\vec{q}(x,y) = \Gamma(x-x_1, y-y_1)\, \vec{Z} \tag{2.83a}$$

Thus the fundamental tensor Γ maps the properly defined singular force vector $\vec{X}$ of strength $\vec{Z}$ at a point (x_1, y_1) into the velocity vector $\vec{q}$ at another point (x,y). The relationship (2.83a) can be written out as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \tag{2.83b}$$

or

$$u = \Gamma_{11} Z_1 + \Gamma_{12} Z_2 \tag{2.83c}$$

$$v = \Gamma_{21} Z_1 + \Gamma_{22} Z_2 \tag{2.83d}$$

For our purposes it is sufficient to consider the flow field generated by a singular force of unit strength located at the origin $(x_1=0, y_1=0)$ and directed along the negative x-axis $(Z_1=-1,\ Z_2=0)$. Then the velocity field $\vec{q}$ is simply expressed by

$$\vec{q} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Gamma_{11} \\ -\Gamma_{21} \end{pmatrix} \tag{2.84}$$

Then the entire flow field is given by the first column of the fundamental tensor, and from Appendix D this may be written out as follows:

$$\begin{pmatrix} \Gamma_{11} \\ \Gamma_{21} \end{pmatrix} = \begin{pmatrix} \Gamma_{o_{11}} \\ \Gamma_{o_{21}} \end{pmatrix} + \begin{pmatrix} \Gamma_{c_{11}} \\ \Gamma_{c_{21}} \end{pmatrix} = \begin{pmatrix} \frac{\partial I}{\partial x} \\ \frac{\partial I}{\partial y} \end{pmatrix} + \frac{1}{2\pi\nu} e^{\frac{Ux}{2\nu}} K_o\left(\frac{Ur}{2\nu}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \Gamma_{c_{11}} \\ \Gamma_{c_{21}} \end{pmatrix} \qquad \text{(2.85)}$$

where

Γ_o = fundamental tensor for an incompressible fluid

Γ_c = fundamental tensor correction to be added for a compressible fluid

$$I = -\frac{1}{2\pi U} \left\{ \log r + e^{\frac{Ux}{2\nu}} K_o\left(\frac{Ur}{2\nu}\right) \right\} \qquad \text{(2.85a)}$$

The compressible correction may be written out as

$$\begin{pmatrix} \Gamma_{c_{11}} \\ \Gamma_{c_{21}} \end{pmatrix} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\beta x}}{i\beta U} \begin{pmatrix} -\beta^2 \\ i\beta \frac{\partial}{\partial y} \end{pmatrix} \left\{ \gamma_{\sqrt{\frac{k}{a}}\,\tau} - \gamma_o \right\} d\beta \qquad \text{(2.86a)}$$

where

$$\gamma_{\sqrt{\frac{k}{a}}\,\tau} = \frac{1}{2} \sqrt{\frac{1+\left(\frac{4}{3}\frac{\nu}{c}\right)^2 M^2 \beta^2}{\beta^2(1-M^2)+\left(\frac{4}{3}\frac{\nu}{c}\right)^2 M^2 \beta^4 + i\left(\frac{4}{3}\frac{\nu}{c}\right) M^3 \beta^3}} \exp\left[-|y| \sqrt{\frac{\beta^2(1-M^2)+\left(\frac{4}{3}\frac{\nu}{c}\right)^2 M^2 \beta^4 + i\left(\frac{4}{3}\frac{\nu}{c}\right) M^3 \beta^3}{1+\left(\frac{4}{3}\frac{\nu}{c}\right)^2 M^2 \beta^2}}\right]$$

$$\mathrm{Re}\,\sqrt{\quad} > 0 \qquad \text{(2.96b)}$$

$$\gamma_o = \frac{1}{2|\beta|} e^{-|y||\beta|} \qquad \text{(2.86c)}$$

The pressure field, p, is also of interest and is simply related to the condensation s by $p = \gamma s$. A fundamental solution vector

$$\vec{S}(x-\xi, y-\eta) = (S_1, S_2) \qquad \text{(2.87)}$$

is defined by

$$s(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\vec{S}(x-\xi,y-\eta)\cdot\vec{X}(\xi,\eta)\,d\xi\,d\eta \tag{2.88}$$

Thus for the singular shearing force the condensation field is given by

$$s(x,y)=-S_1(x,y) \tag{2.89}$$

The results of Appendix D indicate that

$$S_1(x,y)=\frac{-1}{2\pi c^2}\int_{-\infty}^{\infty}e^{i\beta x}\;i\beta\gamma_{\sqrt{\frac{k^2}{a}}}\,d\beta \tag{2.810}$$

which is proportional to the first part of $\Gamma_{c_{11}}$.

Decomposition of Flow Field. The formula for the fundamental tensor (2.85) shows a convenient division into four parts. Thus the velocity field $\vec{q}$ due to the singular shearing stress can be expressed as the sum of four separate fields. Each of these fields is significant and has a simple physical interpretation.

Writing

$$\vec{q}=\vec{q}_1+\vec{q}_{1c}+\vec{q}_2+\vec{q}_2^{\,*} \tag{2.811}$$

and writing each of the fields separately from (2.85), (2.86) we have:

$$\vec{q}_1=\begin{pmatrix}u_1\\ v_1\end{pmatrix}=\frac{1}{2\pi U}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\log r=\frac{1}{2\pi U}\,\mathrm{grad}\log r \tag{2.812}$$

$$\vec{q}_{1c}=\begin{pmatrix}u_{1c}\\ v_{1c}\end{pmatrix}=\begin{pmatrix}-\Gamma_{c_{11}}\\ -\Gamma_{c_{21}}\end{pmatrix} \tag{2.813}$$

$$\vec{q}_2=\begin{pmatrix}u_2\\ v_2\end{pmatrix}=\frac{-1}{2\pi\nu}\,e^{\frac{Ux}{2\nu}}K_0\!\left(\frac{Ur}{2\nu}\right)\begin{pmatrix}1\\ 0\end{pmatrix} \tag{2.814}$$

$$\vec{q}_2^{\,*}=\begin{pmatrix}u_2^*\\ v_2^*\end{pmatrix}=+\frac{1}{2\pi U}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\left[e^{\frac{Ux}{2\nu}}K_0\!\left(\frac{Ur}{2\nu}\right)\right]=\frac{1}{2\pi U}\,\mathrm{grad}\left[e^{\frac{Ux}{2\nu}}K_0\!\left(\frac{Ur}{2\nu}\right)\right] \tag{2.815}$$

Before filling in the details we can make the following general statements about this division. $\vec{q}_1 + \vec{q}_{1c}$ is the longitudinal part of the full velocity field. (See § 1.5). $\vec{q}_1$ itself is the longitudinal part of the incompressible field and $\vec{q}_{1c}$ represents the correction to it which must be added in the compressible case. This correction term of course vanishes when $c \to \infty$, as can be seen in the formulas for Γ_c (2.86). Similarly the pressure field p is divided into the incompressible field p_1, associated with $\vec{q}_1$, and the compressible correction part p_{1c}. $\vec{q}_2 + \vec{q}_2^{\,*}$ is the transversal part of the full velocity field and has no pressure field associated with it. The transversal wave is itself split into two parts, $\vec{q}_2^{\,*}$ and $\vec{q}_2$. $\vec{q}_2^{\,*}$, expressed as a gradient in (2.815), is irrotational. $\vec{q}_2$ which, as will be shown, corresponds to the boundary layer, has all the vorticity associated with it.

Since a singular flat plate is equivalent to a singular shearing stress, the velocity field as given by (2.812)-(2.815) should satisfy all the conditions on the solution for the singular plate. (See 2.41, 2.42 and the discussion given there). This is easily seen except for the compressible correction term (2.812) which will be analyzed later in this section. The formulas (2.812)-(2.815) will now be discussed in some detail and compared with the results obtained by more intuitive consideration in the preceding discussion of the singular plate.

Vorticity. Comparison with §2.6. The simplest way of seeing the connection between the formulas given by the fundamental solution and the results in §2.6 and §2.7 is to consider the vorticity ω :

$$\omega = \text{curl } \vec{q} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \tag{2.816}$$

curl $(\vec{q}_1 + \vec{q}_{1c})$ should be equal to zero because $\vec{q}_1 + \vec{q}_{1c}$ represents the longitudinal part, and this is verified in the formulas (2.812), (2.813), and (2.86). Furthermore curl $\vec{q}_2^{\,*} = 0$ from (2.815). Thus, for the singular shearing force

$$\omega(x, y) = \text{curl } \vec{q}_2 \tag{2.817}$$

But (2.814) yields

$$u_2 = -\frac{1}{2\pi \nu} e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) \tag{2.818}$$

$$v_2 = 0 \tag{2.819}$$

so that

$$\omega(x, y) = \frac{\partial u_2}{\partial y} = \frac{U}{4\pi \nu^2} e^{\frac{Ux}{2\nu}} \frac{y}{r} K_1\left(\frac{Ur}{2\nu}\right) \tag{2.820}$$

This is, within a constant factor, the vorticity due to the vorticity dipole (2.614). There is actually a close relationship between a singular shearing force and vorticity. The shearing stress is proportional to $\frac{\partial u}{\partial y}$ and this is exactly equal to the vorticity in the case where $\frac{\partial v}{\partial x} = 0$, as on a flat plate. Thus the vorticity field due to a singular shearing force corresponds to that expected for an infinitesimal flat plate, the vorticity dipole field. It can now easily be seen that the determination of the entire flow field from the vorticity is rather complicated since the vorticity comes from only a relatively small part of the full solution.

<u>Comparison with §2.7</u>. The term $\vec{q}_2$ is thus quite separate from the rest of the terms in the solution. Its component in the x-direction u_2 is exactly that previously obtained in §2.7 by picking a singular solution for the u-component of the transversal equation (2.77). It is this u_2 term which is related to the boundary layer, and later on it will be shown that the boundary layer only approximates to u_2. Although u_2 is a solution to the dynamic part of the transversal equations, $\vec{q}_2$ itself is not a transversal wave. Obviously the requirement that $\operatorname{div} \vec{q}_2 = 0$ (1.53b) is violated since $v_2 = 0$ (2.819). In §2.7 an iteration procedure starting with u_2 was discussed. If such a procedure does converge, all transversal waves added after the first step (considering v to be zero in the first step) must add up to $\vec{q}_2^{\,*}$, and all longitudinal waves to $\vec{q}_1 + \vec{q}_{1c}$. Also each additional wave satisfies the boundary conditions at infinity. The introduction of longitudinal waves in the iteration procedure is partially justified by the fact that the transversal part of the full solution does not satisfy the boundary conditions. For example, in the incompressible case $v_{1c} = 0$ and the vertical velocity v is given by

$$v = v_1 + v_2^* \tag{2.821}$$

But from (2.812) and (2.815)

$$v_1 = \frac{1}{2\pi U} \frac{y}{r^2} \tag{2.822}$$

$$v_2^* = + \frac{1}{4\pi \nu} \frac{y}{r} e^{\frac{Ux}{2\nu}} K_1\left(\frac{Ur}{2\nu}\right) \tag{2.823}$$

Thus $v = 0$ for $y = 0$ except at $x = 0$ also. The singularities in v_1 and v_2^* at $x = 0$ just cancel out since from (2.612a)

$$K_1\left(\frac{Uy}{2\nu}\right) \doteq - \frac{2\nu}{Uy} \qquad \text{as} \qquad y \to 0 \tag{2.824}$$

The iteration procedure would be very powerful if V_2^* were obtained immediately from u_1. Then the longitudinal wave would be easily found by a requirement that the singularity in V_2^* at the plate be cancelled. This is clearly not the state of affairs in the previous iteration process. A consideration of (2.86) would indicate the effect of compressibility and the exact form of V_{1c}. This has not yet been carried out but it seems clear that V_{1c} must equal zero at $y=0$.

Incompressible Case. Velocity and Pressure. All the previous considerations about transversal waves are valid independent of compressibility. Neither the condensation s nor the parameter c was mentioned. Now we will discuss the rest of the velocity field and the pressure, turning our attention first to the incompressible case. Examining the fundamental tensor (2.85) and (2.85a) we notice that I is a regular function as $r \to 0$, because the K_0 has a logarithmic singularity also, and that I becomes logarithmically infinite as $r \to \infty$. The components of velocity in the x-direction have the following behavior which is easily obtained from (2.812), (2.814) and (2.815)

$$u = u_1 + u_2 + u_2^* \tag{2.825}$$

$$u_1 = \frac{1}{2\pi U}\,\frac{x}{r^2} \tag{2.826}$$

$$u_2 = -\frac{1}{2\pi \nu}\, e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) \tag{2.827}$$

$$u_2^* = \frac{1}{4\pi\nu}\, e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) + \frac{1}{4\pi\nu}\,\frac{x}{r}\, e^{\frac{Ux}{2\nu}} K_1\left(\frac{Ur}{2\nu}\right) \tag{2.828}$$

so that

$$u(x,y) = \frac{1}{2\pi U}\,\frac{x}{r^2} - \frac{1}{4\pi\nu}\, e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) + \frac{1}{4\pi\nu}\,\frac{x}{r}\, e^{\frac{Ux}{2\nu}} K_1\left(\frac{Ur}{2\nu}\right) \tag{2.829}$$

$\vec{q}_1 = (u_1, v_1)$ is the velocity field due to an incompressible source located at the origin and is the simplest singular solution of the potential equation. At $y=0$ we have

$$u(x,0) = \frac{1}{2\pi U}\frac{1}{x} - \frac{1}{4\pi\nu} e^{\frac{Ux}{2\nu}} K_0\left(\frac{U|x|}{2\nu}\right) + \frac{1}{4\pi\nu} (\operatorname{sign} x)\, e^{\frac{Ux}{2\nu}} K_1\left(\frac{U|x|}{2\nu}\right) \tag{2.830}$$

and using the formulas (2.620a) and (2.621a) we have

$$\begin{aligned} u(x,0) &\doteq \frac{1}{2\pi U}\frac{1}{x} + \frac{1}{4\pi\nu}\left\{\log\frac{U|x|}{4\nu} + r\right\} - \frac{1}{4\pi\nu}(\operatorname{sign} x)\left\{\frac{2\nu}{U|x|}\right\} \\ &\doteq \frac{1}{4\pi\nu}\log|x| \qquad \text{as } x \to 0 \end{aligned} \tag{2.830a}$$

Thus, as was to be expected, the singularity in u due to the potential field is cancelled out by part of $\vec{q}_2^{\,*}$. A logarithmic singularity in u itself remains, however. The pressure field for this incompressible source is also obtained from the solution. For an incompressible fluid ρ = const, so that s should approach zero as $c^2 \to \infty$. This is verified in (2.89) and (2.810). However the perturbation pressure depends on the limit of $c^2 s$ as $s \to \infty$ as can be seen by comparing (1.33) and (1.37a), or from (1.36')

$$p = \gamma s = \frac{c^2 \rho_0}{P_0} s \tag{2.831}$$

Now as $c^2 \to \infty$, (2.86b) becomes

$$\gamma_{\sqrt{\frac{k^2}{a}}} \to \frac{1}{2|\beta|} e^{-|y||\beta|} = \gamma_0 \tag{2.832}$$

so that (2.810) indicates

$$p(x,y) = \frac{i}{4\pi}\frac{\rho_0}{P_0}\int_{-\infty}^{\infty} e^{i\beta x - |y||\beta|} \operatorname{sign}\beta \, d\beta \tag{2.833}$$

$$= -\frac{1}{2\pi}\frac{\rho_0}{P_0}\frac{x}{r^2} \tag{2.833a}$$

Noting (2.826) we see that

$$p = -\frac{U\rho_0}{P_0} u_1 \tag{2.834}$$

The pressure field obeys the suitable form of Bernoulli's law according to our linearization. The uniform flow past a source for a non-viscous incompressible fluid gives a well-known velocity and pressure field. The flow is that past a bluff body extending from a small distance to the left of the origin downstream to $x=\infty$. Thus the pressure field as given by (2.833a) is very reasonable, indicating a compression ahead of the source and an expansion downstream of it. Of course when the pressure field originates from a singular shearing force the streamlines are very different.

The pressure field and the velocity associated with it $(\vec{q}_1, p)$ spread out from the source without any preferred direction. However the disturbances which are propagated by the shearing action of the viscosity $\vec{q}_2, \vec{q}_2^{\,*}$ spread mostly downstream, propagating upstream with very heavy damping. Actually this is a process of diffusion relative to the moving fluid while the pressure field spreads by the usual method of propagation relative to the fluid. These facts are easily seen in (2.826-8).

Correction for Compressibility.

For the compressible case some very approximate results have been obtained in Appendix D. They indicate that there is a qualitative distinction between subsonic and supersonic flow at infinity. If we consider first subsonic flow the very approximate formula for the compressible correction $\Gamma_{c_{11}}$ reads

$$\Gamma_{c_{11}} \cong -\frac{1}{2\pi U}\left\{\frac{\dfrac{x}{\sqrt{1-M^2}}}{x^2+(1-M^2)y^2}-\frac{x}{x^2+y^2}\right\} \tag{2.835}$$

$$\Gamma_{c_{21}} \cong \frac{1}{2\pi U}\left\{\frac{y}{x^2+y^2}-\frac{y\sqrt{1-M^2}}{x^2+(1-M^2)y^2}\right\} \tag{2.836}$$

This formula should be valid for large y. Remembering that (u_{1c}, v_{1c}) are given by $(\Gamma_{c_{11}}, -\Gamma_{c_{21}})$ we see that the velocity components of the longitudinal wave are given by

$$u_\ell = u_1 + u_{1c} = \frac{1}{2\pi U} \frac{1}{\sqrt{1-M^2}} \frac{x}{x^2 + (1-M^2) y^2} \tag{2.836}$$

$$v_\ell = v_1 + v_{1c} = \frac{1}{2\pi U} \frac{y\sqrt{1-M^2}}{x^2 + (1-M^2) y^2} \tag{2.837}$$

or

$$\vec{q}_\ell = \frac{1}{2\pi U} \frac{1}{(1-M^2)^{1/2}} \operatorname{grad} \log \sqrt{x^2 + (1-M^2) y^2} \tag{2.838}$$

This indicates that far away from the singularity the effect of compressibility is to modify the flow field by the usual Prandtl rule. The incompressible source of $\vec{q}_1$ is replaced by the corresponding linearized compressible source. We have also the formula for the condensation

$$s = -s_1 = \frac{-1}{2\pi c^2} \frac{1}{\sqrt{1-M^2}} \frac{x}{x^2 + (1-M^2) y^2} \tag{2.839}$$

so that the pressure

$$p_c = \gamma s = \frac{-\rho_0}{2\pi P_0} \frac{x}{x^2 + (1-M^2) y^2} = -\frac{U\rho_0}{P_0} u_\ell \tag{2.840}$$

These approximate formulas have the following defects. They do not involve the viscosity ν and they are certainly not valid for $M \cong 1$. The effect of viscosity on longitudinal waves has been discussed in §§ 2.2 and 2.5 where it was shown that the waves are qualitatively unchanged. The second defect is one typical of all linearized theories near $M=1$. However, as $M \to 0$ (incompressible case) the formulas approach the proper limiting values

$$M \longrightarrow 0 \qquad \vec{q}_\ell \to \vec{q}_1 \tag{2.841}$$

$$M \longrightarrow 0 \qquad p_c \to p_1 \tag{2.842}$$

For the supersonic case $M>1$ the analysis has to be made a little more carefully but similar results are obtained in Appendix D. (Part f). The results valid for large $|y|$ are

$$\Gamma_{c_{11}} \cong \frac{1}{2\pi U}\left\{ \frac{1}{M^{3/2}} \sqrt{\frac{\pi}{\sqrt{M^2-1}\ \frac{8\nu}{3c}|y|}}\ e^{-\frac{\sqrt{M^2-1}\,(x-\sqrt{M^2-1}\,|y|)^2}{\frac{8}{3}\frac{\nu|y|}{c}M^3}} + \frac{x}{x^2+y^2} \right\} \tag{2.843}$$

$$\Gamma_{c_{21}} \cong \frac{-\operatorname{sign} y}{2\pi U}\left\{ \frac{1}{M^{3/2}} \sqrt{\frac{\pi\sqrt{M^2-1}}{\frac{8\nu}{3c}|y|}}\ e^{-\frac{\sqrt{M^2-1}\,(x-\sqrt{M^2-1}\,|y|)^2}{\frac{8}{3}\frac{\nu|y|}{c}M^3}} \right\} + \frac{y}{x^2+y^2}\cdot\frac{1}{2\pi U} \tag{2.844}$$

As before, the effect of the incompressible source is cancelled out and the velocity components and condensation of the longitudinal wave can be written

$$u_\ell = u_1 + u_{1c} = \frac{-1}{2\pi U}\,\frac{1}{M^{3/2}}\,\frac{1}{(M^2-1)^{1/4}} \sqrt{\frac{\pi}{\frac{8}{3}\frac{\nu|y|}{c}}}\ e^{-\frac{\sqrt{M^2-1}\,(x-\sqrt{M^2-1}\,|y|)^2}{\frac{8}{3}\frac{\nu|y|}{c}M^3}} \tag{2.845}$$

$$v_\ell = v_1 + v_c = \frac{+\operatorname{sign} y}{2\pi U}\cdot\frac{1}{M^{3/2}}(M^2-1)^{1/4} \sqrt{\frac{\pi}{\frac{8}{3}\frac{\nu}{c}|y|}}\ e^{-\frac{\sqrt{M^2-1}\,(x-\sqrt{M^2-1}\,|y|)^2}{\frac{8}{3}\frac{\nu|y|}{c}M^3}} \tag{2.846}$$

$$s = -s_1 = \frac{1}{2\pi c^2 M^{3/2}}\,\frac{1}{(M^2-1)^{1/4}} \sqrt{\frac{\pi}{\frac{8}{3}\frac{\nu}{c}|y|}}\ e^{-\frac{\sqrt{M^2-1}\,(x-\sqrt{M^2-1}\,|y|)^2}{\frac{8}{3}\frac{\nu|y|}{c}M^3}} \tag{2.847}$$

The behavior of all these quantities is the same. There is an exponential diffusion about the hyperbolic characteristics extending downstream of the origin, $x-\sqrt{M^2-1}\,|y|=0$, and they all reach their maximum on these lines. The condensation is always positive so that first there is a compression wave followed by expansion behind the peak. Similarly the effect of slowing down ($u_\ell<0$) spreads to the entire field but is concentrated mostly along the hyperbolic characteristics. There is a damping as $\frac{1}{\sqrt{|y|}}$ along the characteristic. The behavior of the solution as $\nu\to 0$ can be considered. For small ν the disturbances are concentrated closer to the characteristics. When $\nu = 0$ the disturbances are zero everywhere except on the characteristics $x - \sqrt{M^2-1}\,|y| = 0$. On the characteristics the values are infinite, typical of the wave equation. The diffuse disturbance coalesces to a sharp wave along the characteristics as the viscosity decreases. However, even when the viscosity is present we notice that

$$\frac{v_\ell}{u_\ell} = -\operatorname{sign} y \cdot \sqrt{M^2-1} \qquad (2.848)$$

This means that, just as for a non-viscous Mach wave, the resultant perturbation of the longitudinal wave is normal to the Mach wave. The sharp wave obtained by this procedure leaves no disturbance behind it. The behavior is that of a source in non-viscous supersonic flow and is best thought of as the limit of the flow past a simple wedge of infinitesimal length. Thus, asymptotically, the infinitesimal flat plate in a viscous supersonic flow develops the same type of pressure field as an infinitesimal wedge in non-viscous supersonic flow plus the effects of diffusion. Finally, it can be noticed that although (2.846) is supposed to be valid for large $|y|$ it satisfies the correct type of boundary conditions at $y=0$. $v_\ell = 0$ at $y=0$ except when $x=0$ where there is a singularity.

A few general remarks about this section can be made. It should be noted that the expressions for the velocity components all appear to have the dimensions $(\text{viscosity})^{-1}$, i.e. $(\text{length})^{-2}$ (time). This depends on the fact that they actually are components of the fundamental tensor. It can be seen from the definition of the fundamental tensor that its components must have the dimension velocity divided by the dimension of the quantity $\iint \vec{X}\,dx\,dy$ which is (length)^3 (time)$^{-2}$. In the calculations it is assumed that $\vec{X}$ is a Dirac function such that the value of the integral is the vector $\binom{1}{0}$. Thus if it is taken into account that the number 1 in this vector actually is dimensional, the dimensions of the velocity vector will check.

Furthermore, it should be noted that the values of the velocity components do not tend to zero when $\mu \to 0$. This result is not paradoxical, as may be seen by the following argument.

Consider as an approximation to the singular shearing stress a flat plate of very small length ℓ with the condition that on the flat plate $u = -u_0$, $u_0 > 0$. Also assume for convenience that the free-stream density is unity. Then for fixed values of μ and ℓ, the given retarding action u_0 of the flat plate may always be adjusted so that the total force exerted by the plate on the fluid is unity. If now μ tends to zero while ℓ and u_0 are kept fixed, the perturbation field generated by the plate will tend to zero. At the same time the force will tend to zero. If one wants to maintain a force of unit strength as $\mu \to 0$ the retarding action u_0 has to become infinite as $\frac{1}{\mu}$. Thus the velocity at the plate becomes infinite while the length of the plate retains its fixed finite value ℓ. It is the combination of these two circumstances that leads to singularities so strong that the field does not vanish as $\mu \to 0$. It is thus seen that in considering such a limiting case, it is important to distinguish between the problem where the velocity at the plate is prescribed and the problem where the shearing stress is prescribed. The difference between the two types of problems will be further discussed in Chapter 3.

In the discussion in this section, it has always been assumed that $\vec{X}$ is parallel to the free-stream direction. The second column of the fundamental tensor describes the velocity field due to a singular shearing stress at right angles to the free-stream direction. This flow field is different from the solution previously considered. This is so because the x-axis is a preferred direction and the problem does not have rotational symmetry.

The expressions for the fundamental solution should be analyzed further. In particular, it is desirable to analyze the singularity at the plate and obtain approximate formulas for the neighborhood of the plate.

<u>Non-Stationary Case.</u> A short discussion of the two-dimensional non-stationary flow will now be given. In particular, it will be shown that the intuitive considerations of §2.6 and §2.7 are essentially correct. Since the reasoning is very similar to that used for the stationary case, only a few details will be given.

The symbols $\vec{X}$, Γ and Γ_{ij} will be used as in the stationary case. However, in the present case, $\vec{X}$ depends on time also. The fundamental solution gives the velocity field due to a force vector $\vec{X}$ which is a Dirac function in both space and time. This means that $\vec{X}$ becomes singular in such a way that the total impulse $\rho\iiint \vec{X}\,dx\,dy\,dt$ stays finite. In the present case, there is no preferred direction. We may assume that $\vec{X}$ is directed along the nagative x-axis without any loss of generality. The flow field for a force in any other direction can easily be obtained by a rotation which makes the x-axis coincide with that direction.

The fundamental tensor is derived in non-dimensional coordinates in Appendix D (Equation D76). The change to dimensional coordinates is made by replacing t, x, y and σ in (D76) by $\frac{3c^2t}{4\nu}$, $\frac{3cx}{4\nu}$, $\frac{3cy}{4\nu}$, $\frac{4\nu\sigma}{3c^2}$ respectively (cf. 2.27). Furthermore, a unit impulse in dimensional coordinates corresponds to a value of $\frac{4\nu}{3c^2}$ for the x-component of $\iiint \vec{X}\, dx\, dy\, dt$, if $\vec{X}$ is the force term in the non-dimensional equation. Hence, in addition to the substitution mentioned, the expressions in (D76) should be multiplied by $\frac{3c^2}{4\nu}$. A similar transformation should be carried out in the formula for the pressure (D77).

Thus the flow field due to a singular shearing stress of unit impulse located at the origin and directed along the negative x-axis is, in dimensional coordinates

$$\vec{g}(x,y,t) = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Gamma_{11}(r,t) \\ -\Gamma_{21}(r,t) \end{pmatrix} = -\begin{pmatrix} \Gamma_{11}^{(1)} \\ \Gamma_{21}^{(1)} \end{pmatrix} - \begin{pmatrix} \Gamma_{11}^{(2)} \\ \Gamma_{21}^{(2)} \end{pmatrix} \tag{2.849}$$

The longitudinal part is defined by

$$\begin{pmatrix} \Gamma_{11}^{(1)} \\ \Gamma_{21}^{(1)} \end{pmatrix} = \frac{4\nu}{3} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \frac{1}{2\pi i} \frac{\partial}{\partial x} \int_{-i\infty}^{i\infty} e^{\sigma t} \frac{1}{2\pi\sigma} K_0\left(\frac{\sigma r}{\sqrt{c^2 + \frac{4}{3}\nu\sigma}} \right) d\sigma \tag{2.850}$$

and the transversal part by

$$\begin{pmatrix} \Gamma_{11}^{(2)} \\ \Gamma_{21}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\partial I}{\partial y} \\ \frac{-\partial I}{\partial x} \end{pmatrix} \qquad (2.851a)$$

where

$$I = -\frac{1}{2\pi} \frac{\partial}{\partial y} \int_r^\infty \frac{1}{\xi} e^{-\frac{\xi^2}{4\nu t}} d\xi = -\frac{1}{2\pi} \frac{4\nu y}{3r^2} e^{-\frac{r^2}{4\nu t}} \qquad (2.851b)$$

The pressure field associated with the longitudinal wave is (cf. D77 and 2.831)

$$c^2 s = \left(\frac{4\nu}{3}\right)^2 \frac{\partial}{\partial x} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma t} \frac{1}{2\pi \sqrt{c^2 + \frac{4\nu\sigma}{3}}} K_0\left(\frac{\sigma r}{\sqrt{\sigma^2 + \frac{4\nu\sigma}{3}}}\right) d\sigma \qquad (2.852)$$

A comparison of (2.851) and (2.612) shows that the transversal part is, within a constant factor, the transversal flow field associated with the vorticity dipole (2.611). Note that within a constant factor $I = \frac{\partial}{\partial y} \psi_0$, where ψ_0 is defined by (2.69). In particular the vorticity field generated by the singular shearing stress is that of the vorticity dipole. The longitudinal part gives the compensating longitudinal wave discussed on p. 75. It is related to the longitudinal waves discussed in §2.5 through simple differentiations. Thus the intuitive approach to the problem of the singular flat plate in §2.6 has been justified.

As in the stationary case, the iteration procedure (§2.7, p. 80) does not show a simple connection with the fundamental solution. The first step (2.72) of this iteration does not seem to be related to the transversal wave (2.851) in a simple way.

It can be seen that as c becomes infinite, the expressions for velocity and pressure reduce to those given by Oseen (Ref. 21) for the incompressible case. However, this question and other problems regarding the nature of the solutions (2.850) and (2.851) will not be discussed further in this report.

3. BOUNDARY-VALUE PROBLEMS

§3.1 Boundary Conditions. Symmetries.

Specific solutions of boundary-value problems which would show detailed agreement with the behavior of a real fluid flowing past an object cannot be expected at the present stage. The linearized equations are not adequate for a description of the entire flow field, even if they could be completely solved. A difficulty is introduced by the boundary conditions, even for a slender body whose local angle of attack is small everywhere. The no-slip condition requires that the magnitude of the "perturbation" velocity at the surface of the body be equal to that of the free-stream velocity. This violates the condition for linearization. The difficulty is especially apparent in supersonic flow. Hence, for the problems solved here we shall require the tangential velocity at the body to be given by $(U+u_0, v_0)$ where u_0 and v_0 are prescribed functions, everywhere small compared with U. The justification for this is discussed in the Introduction (p. 7). We shall restrict ourselves to treating two-dimensional stationary flow past a half-infinite or finite flat plate at zero angle of attack.

The boundary and symmetry conditions for such a case will now be discussed. Far upstream $(x \to -\infty)$ the flow should tend to a uniform flow parallel to the x-axis. Also, all the disturbances should vanish as $y \to \pm\infty$ for a fixed x. On the plate itself we require that $u = u_0$, and $v = 0$. We assume that the plate is located on the x-axis. Then, under the further assumption that the flow is determined uniquely by the prescription of these conditions alone, certain conditions of symmetry can be deduced. The purpose of imposing conditions of symmetry is to enable us to solve the problem in a half-plane (say $y \geqslant 0$). Under the prescribed conditions the flow field must be symmetric about the x-axis. If it were not, a different solution could be constructed by reflection about $y=0$.

Next we discuss the symmetry of u_y, which is related to the shearing stress at any point. The differential equations (1.43) which describe the local equilibrium of forces and accelerations are valid, with the external force $\vec{X}=0$, in the plane minus the slit corresponding to the flat plate. Thus in any region of the plane (such as the rectangles B or C in Fig. 3.1) which contains no part of the plate, hydrodynamic stresses alone will balance the inertia forces. This is not true for the rectangle A. In this region the plate exerts a force on the fluid whose magnitude is equal to the skin friction. This has an important consequence for the derivative u_y. It follows from symmetry that u_y is antisymmetric with respect to the x-axis. Hence on the x-axis u_y is either equal to zero or discontinuous. The

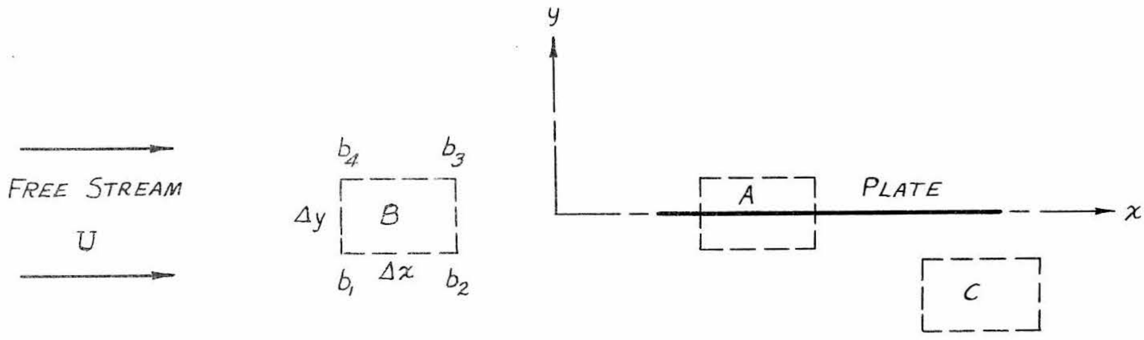

Fig. 3.1

former alternative occurs off the plate and the latter at the plate. Let us consider the rectangle B. The shearing force of the fluid outside B on $b_4 b_3$ is $\mu u_y \Delta x$. If B is symmetrically located with respect to the x-axis the force on $b_1 b_2$ is the same. Now if u_y approached a limit different from zero as Δy approached zero, there would result an infinite force per unit area on the x-axis. Excluding the possibility that the other hydrodynamical forces also become infinite, this force per unit area $2\frac{\mu u_y}{\Delta y}$ requires an external force to balance it. Such a force is provided by the plate in the rectangle A, but not in B. Hence we conclude that, on the x-axis, $u_y = 0$ off the plate whereas we expect it to be different from zero on the plate.

Since both u_y and v_x are odd functions of y, the same must be true of the vorticity $\omega = u_y - v_x$. On the x-axis $v_x = 0$; hence $\omega(x,0) = u_y(x,0)$.

Summarizing, we obtain the following table of conditions for a flat plate along the x-axis:

Quantity	On Plate	Off Plate on x-axis	Symmetry		
u	u_0	Unknown	Sym	(a)	
u_y	Unknown	0	Antisym	(b)	(3.11)
v	0	0	Antisym	(c)	
ω	u_y	0	Antisym	(d)	

In addition, we have the condition that disturbances vanish at large distances from the plate. It is used in the following form:

$$u(-\infty, y) = v(-\infty, y) = 0 \tag{3.12a}$$

$$u(x, \infty) = v(x, \infty) = 0 \tag{3.12b}$$

Thus, the mathematical formulation of the problem is given by equations (1.43) in a two-dimensional field with $\vec{X}=0$, and boundary conditions (3.11) and (3.12). We are not concerned, here, with the question of determining which of the conditions in the table are redundant.

Note that u is prescribed on part of the x-axis and u_y is prescribed on the complement. The boundary conditions are thus of a mixed type. This is similar to the problem of a lifting wing of zero thickness in ordinary wing theory where the downwash (φ_y) is prescribed on the wing and the pressure $(\sim \varphi_x)$ is known to be zero off the wing. The mixed boundary condition explains why the problem leads to an integral equation, rather than to a representation as an explicit integral.

The remainder of this chapter is devoted to a study of the flat plate problem, particularly when the plate extends to infinity along the positive x-axis. Several methods of attack on the problem are discussed in the following sections. None of these lead to a complete solution of the problem; but it is hoped that they bring a contribution capable of further development. The first two methods are directly related to Chapter 2.

§3.2 Integral Equations Based on the Fundamental Solutions

In this section it will be shown that any finite flat plate may be considered as a superposition of singular flat plates. The problem of finding the distribution of singular flat plates leads to an integral equation. A solution of this integral equation is not attempted; however, the nature of the problem is illustrated by solving a similar integral equation resulting from the boundary-layer equation.

First let us consider a simple example of superposition. Let the flat plate described in the previous section extend between $x=a$ and $x=b$ and let u_o be a function $u_o^{(1)}(x)$ defined for $a \leq x \leq b$. Consider also a second plate between b and c where the u velocity is prescribed to be $u_o^{(2)}(x)$ defined for $b \leq x \leq c$. Now superimpose the solutions for the two cases. It is seen from the table of boundary conditions and symmetries (3.11) and (3.12) that a solution for a plate between a and c is obtained. Note in particular that the condition on u_y will be satisfied since it is homogeneous. However, the u distribution on the combined plate cannot be determined from the boundary conditions on the component plates only. For $a \leq x \leq b$, u is $u_o^{(1)}(x)$ plus whatever upstream disturbance in u is caused by the second plate. Similarly, for $b \leq x \leq c$, u is $u_o^{(2)}(x)$ plus the downstream disturbance of the first plate. These upstream and downstream disturbances are in general not zero (see, for instance, p. 78).

We can now extend this idea to a superposition of "infinitely short" plates, namely the singular plates discussed in the previous chapter and in Appendix D. The velocity field due to such a singular plate is represented by the first column of the fundamental solution Γ :

$$u = -\Gamma_{11}(x-\xi, y) \tag{3.21a}$$

$$v = -\Gamma_{21}(x-\xi, y) \tag{3.21b}$$

as given by (2.84), (2.85), and (2.86). This solution satisfies all the symmetry and boundary conditions of (3.11) and (3.12). The value of u is of course infinite on the plate. Still, in a sense it may be varied by multiplying the solution by a strength factor f, which corresponds to the source strength in an ordinary hydrodynamical problem. If now a distribution of singular flat plates, each of strength $-f(\xi)\,d\xi$, is placed along the x-axis between a and b, the following flow field results:

$$u(x,y) = \int_a^b f(\xi)\,\Gamma_{11}(x-\xi, y)\,d\xi \tag{3.22a}$$

$$v(x,y) = \int_a^b f(\xi)\,\Gamma_{21}(x-\xi, y)\,d\xi \tag{3.22b}$$

This solution satisfies all the conditions for a flat plate extending between a and b. The u velocity on the plate is then

$$u_o(x) = \int_a^b f(\xi)\,\Gamma_{11}(x-\xi, 0)\,d\xi \tag{3.23}$$

When $f(\xi)$ is prescribed, (3.22) determines u. Various solutions representing flow past a flat plate may be constructed by varying f. The velocity at the plate is now given by (3.23). Prescribing f actually amounts to prescribing the force distribution (skin friction) at the flat plate. However, in the boundary value problem described in §3.1 $u_o(x)$ rather than f is given. (3.23) represents the fundamental integral equation for $f(\xi)$. Once this is solved, the value for $f(\xi)$ may be substituted into (3.22), which then gives the solution in the form of explicit integrals. In particular, we are interested in the case $u_o(x) =$ constant, $a = 0$, $b = \infty$. This is referred to as the half-infinite flat plate.

An attempt to solve (3.23) will not be made here. Instead a simpler case is treated, where the equation has the same structure, but where the analysis is simpler. We investigate the problem of the half-infinite flat plate, using the boundary-layer equation (1.59) for u, with the boundary condition on the plate $u = u_o$ = constant and neglecting the v component completely. The fundamental solution is given by (2.79a). For $y = 0$ it is simply

$$u(x,0) = \frac{1}{\sqrt{x}} \qquad x > 0 \tag{3.24a}$$

$$u(x,0) = 0 \qquad x < 0 \tag{3.24b}$$

Then, the integral equation corresponding to (3.23) is Abel's equation:

$$u_o = \int_0^x \frac{f(\xi)}{\sqrt{x-\xi}}\, d\xi \tag{3.25}$$

In this particular case, the equation is of the Volterra type: the upper limit of the integral is x rather than ∞. This is so because the fundamental solution (2.79) indicates no propagation of disturbances upstream. Equation (3.25) is easily solved:

$$f(\xi) = \frac{-2u_o}{\sqrt{\pi\xi}} \tag{3.26}$$

The solution of the flat plate problem is therefore:

$$u(x,y) = -\frac{2u_o}{\pi} \int_0^x e^{-\frac{Uy^2}{4\nu(x-\xi)}} \frac{d\xi}{\sqrt{\xi(x-\xi)}} \qquad x > 0 \tag{3.27a}$$

$$u(x,y) = 0 \qquad x < 0 \tag{3.27b}$$

This integral can be evaluated, and the result is identical with that obtained directly in the following section (3.32).

Returning to the original problem based on the complete linearized equations (1.43) rather than the boundary-layer equations, we find the analytical work enormously complicated because of the structure of the kernels Γ_{11} and Γ_{21}. Even the incompressible case seems very difficult. However, it seems reasonable to hope that further work along these lines will lead to interesting results. Asymptotic evaluations of the solution may be possible. Various iteration procedures should also be tried. It is also of interest to prescribe the source strength $f(\xi)$ and investigate the resulting values of $u(x,0)$ on the plate.

§3.3 Solutions of Boundary-Layer Type. Iteration Procedure:

The ideas presented in the present section are closely related to those of Ref. 50, Chap. IIE, even though the mathematical technique and the equation studied are quite different.

A method for treating the flat plate problem similar to the one described in §2.7 will be discussed. As in that case, the method will lead to an iteration procedure which it does not seem possible to carry out; it will, however, contribute to a qualitative understanding of the limitations of boundary-layer theory and of the interaction between transversal and longitudinal waves.

If standard boundary-layer theory is applied to the linearized equations (1.33-1.35) with the boundary conditions of the flat plate problem, the following equation is obtained (see 1.59 and the references given there)

$$\partial u_{yy} - U u_x = 0 \tag{3.31}$$

This equation is the equation for a transversal wave in u (1.57a) with the term νu_{xx} left out. Since it is a simple heat equation, the solution for the flat plate problem is obtained immediately

$$u = u_0 \operatorname{erfc} \left(\frac{U y^2}{4 \nu x} \right)^{\frac{1}{2}} \qquad \text{for} \qquad x > 0 \qquad (3.32a)$$

$$u = 0 \qquad \text{for} \qquad x < 0 \qquad (3.32b)$$

(3.32) is Rayleigh's solution for the linearized boundary-layer problem (Ref. 43). Note that the two parts of the function join smoothly along the y-axis, the function given by (3.32) has all its partial derivatives equal to zero for $x=0$.

If one tries to compute v from the continuity equation $u_x + v_y = 0$, one sees immediately that the two conditions $v=0$ at $y=0$ and $y=\infty$ may not be satisfied simultaneously. This is the same difficulty which was encountered in §2.7. However, instead of discussing this difficulty for the simple boundary-layer equation (3.31), we shall construct a solution for the full transversal equation for u and then discuss a similar problem.

The procedure is as follows. The equation for the x-component of the transversal wave is rewritten in parabolic coordinates. In the resulting equation a natural approximation of the boundary-layer type is made. With the aid of similarity considerations a solution for this simplified equation is obtained. It then turns out that this solution actually satisfies the full transversal equation: the term that had been left out in the approximation is identically zero.

The parabolic coordinates ξ, η are introduced by

$$x = \xi^2 - \eta^2, \quad y = 2\xi\eta, \quad \eta \geq 0 \qquad \xi \text{ and } y \text{ have the same sign} \quad (3.33a)$$

The inverse formulas are

$$\xi^2 = \frac{1}{2}\left(\sqrt{x^2+y^2}+x\right), \qquad \eta^2 = \frac{1}{2}\left(\sqrt{x^2+y^2}-x\right) \tag{3.33b}$$

Here the plane is cut along the plate $(y=0,\ x>0)$. ξ is discontinuous across this slit, being positive above and negative below. The plate is represented by $\eta = 0$; the half line $y=0$, $x<0$ is represented by $\xi = 0$. The following expansion, valid for $y<x$ will be needed

$$\eta^2 = \frac{1}{4}\,\frac{y^2}{x} + x \cdot \text{(fourth order terms in } \frac{y}{x}) \tag{3.33c}$$

In these coordinates equation (1.57a) becomes

$$\xi u_\xi - \eta u_\eta = \frac{\nu}{2U}\,\left(u_{\xi\xi} + u_{\eta\eta}\right) \tag{3.34}$$

and the boundary conditions for the flat plate problem are

$$u(\xi, 0) = u_0 \tag{3.35a}$$

$$u_\xi(0,\eta) = 0 \tag{3.35b}$$

$$u(\xi, \infty) = 0 \tag{3.35c}$$

The previous boundary-layer assumption was that the most rapid changes take place in directions perpendicular to the plate. This is obviously not very satisfactory for regions near the nose of the plate. Instead, the assumption will now be made that the most rapid changes take place in directions perpendicular to the lines η = constant. This agrees with the previous assumption far downstream (cf. 3.33c) and seems better suited for the region near the nose. If this assumption is made the basis for a boundary-layer approximation (cf. Ref. 52), the term $u_{\xi\xi}$ drops out and one obtains

$$\xi u_{\xi} - \eta u_{\eta} = \frac{\nu}{2U} u_{\eta\eta} \tag{3.36}$$

But this equation is invariant under a linear stretching of the ξ-coordinate, i.e. the mapping $\eta \rightarrow \eta$, $\xi \rightarrow k\xi$, k = constant. If $u(\xi,\eta)$ is a solution then so is $u(k\xi,\eta)$. If the problem has a unique solution, this must then be independent of ξ. Hence, it is natural to try a solution of the form $u = u(\eta)$. Such a solution is easily obtained. It is

$$u(\eta) = \frac{2u_0}{\sqrt{\pi}} \int_{\eta/\sqrt{\frac{\nu}{U}}}^{\infty} e^{-\sigma^2} d\sigma = u_0 \operatorname{erfc}\left(\eta\sqrt{\frac{U}{\nu}}\right) \tag{3.37}$$

or

$$u(x,y) = u_0 \operatorname{erfc} \sqrt{\frac{U}{2\nu}\left(\sqrt{x^2+y^2} - x\right)} \tag{3.37'}$$

Since this function is actually independent of ξ, $u_{\xi\xi}$ is identically zero. Thus since (3.37) is a solution of (3.36) it is also a solution of the original equation (3.34).

It might be expected that (3.37) should agree with Rayleigh's boundary-layer solution (3.32) for points far downstream. Actually, it follows from (3.33c) that

$$\frac{U\eta^2}{\nu} \approx \frac{Uy^2}{4\nu x} \qquad \text{for} \quad y << x$$

which proves the preceding statement and also makes it more precise. Contrary to Rayleigh's solution, (3.37) shows upstream disturbances. This should be considered as a definite improvement since we know from § 2.6 that vorticity spreads upstream although with a heavy exponential damping. The actual damping in the present case may be estimated from (3.37) and the asymptotic formulas (Ref. 12, Appendix II):

$$\operatorname{erfc} z = 1 - \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3}\right) + \dots \quad \text{for } |z| \text{ small} \tag{3.38a}$$

$$\operatorname{erfc} z = \frac{1}{\sqrt{\pi}} e^{-z^2}\left(\frac{1}{z} - \frac{1}{2z^3}\right) + \dots \text{ for } |z| \text{ large} \tag{3.38b}$$

In analogy with the procedure in §2.7 we now try to see if (3.37) may be made part of a complete solution which also includes v. If this complete solution were a transversal wave, v would be determined from u with the aid of the simple continuity equation $u_x + v_y = 0$. In parabolic coordinates this equation reads

$$\xi u_\xi - \eta u_\eta - \xi v_\eta + \eta v_\xi = 0 \tag{3.39}$$

However, the same difficulty that was encountered in §2.7 arises here again. (3.39) is a first-order equation and v has to satisfy two boundary conditions corresponding to (3.11) and (3.12):

$$v = 0 \quad \text{for} \quad \eta = 0 \tag{3.310a}$$

$$v = 0 \quad \text{for} \quad \eta = \infty \tag{3.310b}$$

These may not be satisfied simultaneously. This proves that the boundary conditions for a flat plate may not be satisfied by a transversal wave only; thus longitudinal pressure waves will occur also. Some conception of these pressure waves may be obtained by using the solution obtained as a first step in an iteration procedure analogous to the one described in §2.7. Thus we determine v from the solution for u (3.37) and the simple continuity equation (3.39) by only satisfying the boundary condition for $\eta = \infty$ (3.310b). In boundary-layer theory one usually retains the condition $v=0$ at the plate and relaxes the condition at infinity instead. However, the present approach is more consistent with the idea of the bulging of the boundary layer. The following expression for v is obtained:

$$v = u_o e^{\frac{(\xi^2-\eta^2)U}{\nu}} \operatorname{erfc} \xi\sqrt{\frac{U}{\nu}} \qquad \text{for } \xi > 0 \tag{3.311}$$

$$v = u_o e^{\frac{xU}{\nu}} \operatorname{erfc} \left\{ \frac{U}{2\nu}\left(\sqrt{x^2+y^2}+x\right)\right\}^{\frac{1}{2}} \qquad \text{for } y > 0 \tag{3.311'}$$

Values for ξ and $y<0$ are obtained by the rule of antisymmetry:

$$v(x,y) = -v(x,-y)$$

In particular for $y=0$:

$$v = \pm u_o e^{\frac{Ux}{\nu}} \operatorname{erfc} \sqrt{\frac{Ux}{\nu}} \tag{3.312}$$

where the plus sign applies to the top side of the x-axis and the minus sign to the lower side.

In estimating the values of v, (3.38) should be used. It follows that $v = \pm u_o$ at the origin, decreases exponentially upstream, and decreases downstream as $\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$:

$$v \approx \pm \frac{u_o}{\sqrt{\pi}} \left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}} \qquad \text{for } x \text{ large} \tag{3.313}$$

These estimates should be compared with the corresponding asymptotic formulas for the singu[illegible] flat plate, §§2.6 and 2.7. This shows that condition (3.310a) is not satisfied for $y=0$ and that v actually has a discontinuity here if the condition of antisymmetry is to be maintained. Note that $\frac{u_x}{u_o} > 0$; this, together with the continuity equation, implies that $\frac{v}{u_o}$ decreases monotonely from a positive value at $y=0$

to the value zero at $y=+\infty$. In thinking in intuitive terms we assume that the plate retards the flow, which means that u_o is negative. This will always be understood when terms such as downwash, compression (as opposed to upwash and expansion) are used. Thus (3.312) gives a downwash distribution on top of the plate and upwash distribution on the lower side. The same is true on the x-axis ahead of the plate.

If the formulas obtained are taken as a basis for an iteration procedure, the next step would be to introduce a longitudinal wave which would cancel the v-distribution along the x-axis. This wave would then have upwash on the top side and a corresponding amount of downwash along the lower side. It would thus correspond to longitudinal viscous flow past a symmetrical body extending along the x-axis whose thickness increases monotonely with x. Its slope has a maximum equal to $-u_o$ at $x=0$ and decreases exponentially upstream and as $\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$ downstream. No condition is imposed on the u-velocity of the wave (otherwise one could not obtain a pure longitudinal wave). It follows from §2.5 that in this case viscosity has mainly a dispersive effect and that a qualitative estimate may be made by solving the non-viscous equation $(1-M^2)\varphi_{xx}+\varphi_{yy}=0$. The boundary conditions are given by (3.312) with a reversal of sign. The supersonic case is especially interesting. In this case the horizontal velocity component (denoted by $\bar{u}$) of the compensating non-viscous longitudinal wave is

$$\bar{u}=\frac{u_o}{\sqrt{M^2-1}}\, e^{\frac{U(x-y\sqrt{M^2-1})}{\nu}}\, \operatorname{erfc}\left\{\frac{U}{\nu}\left(x-y\sqrt{M^2-1}\right)\right\}^{\frac{1}{2}} \quad \text{for } y>0 \qquad (3.314)$$

and

$$\bar{u}(x,y) = \bar{u}(x,-y)$$

The longitudinal wave would carry a pressure disturbance which in the non-viscous approximation is equal to $-\rho U\bar{u}$. Thus compression waves are sent out from "the boundary layer" into the "outer fluid". These waves are especially strong near the nose of the plate. "Compression" is meant relative to the undisturbed fluid, in accordance with the terminology of the linearized theory. The maximum compression is at $x=0$; farther downstream the fluid expands relative to the upstream values even though it is in a state of compression compared to the undisturbed flow ($\bar{u}$ is always negative). In the terminology used in non-linear waves one would say that there is a sharp compression just ahead of the nose, followed by a gradual expansion back to the free-stream value of the pressure.

The addition of the longitudinal wave violates the original boundary conditions on u_o on the plate and on u_y ahead of the plate. Thus, even if formula (3.314) and the corresponding formula for $\bar{v}$ were corrected for viscosity, still more solutions would have to be added; there is no reason to believe that the iteration procedure would end after a finite number of steps or even converge. Actually the magnitude of the error introduced in u is of the same order as the original error in v . Neglecting viscosity, on the top side of the plate $\bar{u} = \frac{v}{\sqrt{M^2-1}}$ where v is to be taken from (3.312). Thus after the compensating longitudinal wave has been added, the value of u at the nose is $u_o\left(1+\frac{1}{\sqrt{M^2-1}}\right)$ instead of u_o . Further downstream, the magnitude of the correction $\bar{u}/u_o$ decreases as $\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}$ (see 3.313). Presumably this situation would not be changed in order of magnitude by taking the viscosity of the longitudinal wave into account. This situation might have been improved if the original transversal wave had been determined from a u-distribution smaller than the one prescribed by the boundary conditions.

The method discussed above thus has obvious shortcomings as a process for determining the complete solution. However, it gives a certain insight into the mutual interaction between transversal and longitudinal waves. It is also a logical development of boundary-layer theory and indicates the limitation of this theory.

§3.4 Integral Equations Based on Fourier Analysis

Another set of integral equations, equivalent to that derived in §3.3, is obtained by constructing the solution of the boundary-value problems by means of Fourier integrals.

As an example of this method, consider the problem of steady flow past a semi-infinite flat plate. It was shown (§1.5) that the equations of motion lead to the following split system for the velocity components u, v

$$u = u_1 + u_2 \qquad v = v_1 + v_2 \tag{3.41}$$

$$(M^2-1)\, u_{1_{xx}} - u_{1_{yy}} = \frac{4}{3}\frac{\partial U}{c^2}\left(u_{1_{xxx}} + u_{1_{xyy}}\right) \tag{3.42}$$

$$M u_{2_x} = \frac{\partial}{c}\left(u_{2_{xx}} + u_{2_{yy}}\right) \tag{3.43}$$

with the additional conditions

(3.44a)

which can be used to find v from u (3.44b)

It has also been shown that the boundary conditions are (see §3.1)

$$u(0,x) = u_0 \qquad (0 < x < \infty) \tag{3.45a}$$

$$u_y(0,x) = 0 \qquad (-\infty < x < 0) \tag{3.45b}$$

$$v(0,x) = 0 \tag{3.45c}$$

$$u(-\infty, y) = v(-\infty, y) = 0 \tag{3.45d}$$

$$u(x,\infty) = v(x,\infty) = 0 \tag{3.45e}$$

General solutions of equations (3.42) and (3.43) for $y > 0$ can be constructed by means of the Fourier integral. It is found by seeking a solution of the form $e^{i\lambda x} f(y)$ that a solution of (3.42) is

$$u_1 = e^{i\lambda x} e^{-y\lambda\sqrt{\frac{\frac{4}{3}\frac{\nu M}{C}\lambda i - (M^2-1)}{1+\frac{4}{3}\frac{\nu M}{C}\lambda i}}} \tag{3.46a}$$

Similarly for (3.43)

$$u_2 = e^{i\lambda x} e^{-y\sqrt{\lambda^2 + \frac{MC}{\nu}\lambda i}} \tag{3.46b}$$

These simple solutions are now generalized. Considering λ as a complex variable, one can cut the λ plane to determine the radicals occurring in (3.46) so that their real parts are positive. Thus,

$$\lambda\sqrt{\frac{\frac{4}{3}\frac{\nu M}{C}\lambda i - (M^2-1)}{1+\frac{4}{3}\frac{\nu M}{C}\lambda i}} = \alpha + i\beta \qquad \text{where } \alpha > 0 \tag{3.47a}$$

and

$$\sqrt{\lambda^2 + \frac{MC}{\nu}\lambda i} = \rho + i\sigma \qquad \text{where } \rho > 0 \tag{3.47b}$$

Convergence as $y \longrightarrow +\infty$ is thus insured. One is then led to examine the following expressions as possible solutions

$$u_1 = \int_{-\infty}^{\infty} \tilde{u}_1(\lambda) e^{i\lambda x} e^{-y(\alpha+i\beta)} d\lambda \qquad v_1 = \int_{-\infty}^{\infty} \tilde{v}_1(\lambda) e^{i\lambda x} e^{-y(\alpha+i\beta)} d\lambda \tag{3.48a}$$

$$u_2 = \int_{-\infty}^{\infty} \tilde{u}_2(\lambda) e^{i\lambda x} e^{-y(\rho+i\sigma)} d\lambda \qquad v_2 = \int_{-\infty}^{\infty} \tilde{v}_2(\lambda) e^{i\lambda x} e^{-y(\rho+i\sigma)} d\lambda \tag{3.48b}$$

These integrals, if they exist, satisfy equations (3.42), (3.43) and vanish as $y \longrightarrow \infty$.

To satisfy equations (3.44a, b), one must have:

$$\tilde{v}_1 = \frac{\beta - i\alpha}{\lambda} \tilde{u}_1 \tag{3.49a}$$

$$\tilde{v}_2 = \frac{i\lambda(\rho - i\sigma)}{\rho^2 + \sigma^2} \tilde{u}_2 \tag{3.49b}$$

Boundary condition (3.45c) implies that

$$\tilde{v}_1 + \tilde{v}_2 = 0 \tag{3.410}$$

so that

$$\tilde{u}_2 = \frac{(\alpha\rho - \beta\sigma) + i\,(\beta\rho + \alpha\sigma)}{\lambda^2} \tilde{u}_1 \tag{3.411}$$

One must still determine the function $\tilde{u}_1$, which is defined by the following conditions, obtained from equations (3.45a, b) and (3.411)

$$\int_{-\infty}^{\infty} \tilde{u}_1 \left(1 + \frac{(\alpha\rho - \beta\sigma) + i\,(\beta\rho + \alpha\sigma)}{\lambda^2} \right) e^{i\lambda x} d\lambda = u_0 \qquad (0 < x < \infty) \tag{3.412a}$$

$$\int_{-\infty}^{\infty} \tilde{u}_1 \left\{ (\alpha + i\beta) + \frac{\rho + i\sigma}{\lambda^2} \left[(\alpha\rho - \beta\sigma) + i\,(\beta\rho + \alpha\sigma) \right] \right\} e^{i\lambda x} d\lambda = 0 \qquad (-\infty < x < 0) \tag{3.412b}$$

The system of two integral equations (3.412) defines $\tilde{u}_1$; if the function $\tilde{u}_1(\lambda)$ obtained by solving this system is such that the integrals (3.48) exist, then those integrals give the required solution. The system (3.412a,b) can be reduced to a single inhomogeneous equation of the Wiener-Hopf type, which will be discussed in a forthcoming paper. An explicit asymptotic solution, valid for $x \to +\infty$ can be obtained by methods similar to those used for constructing asymptotic solutions for one-dimensional waves (see §2.2). That solution gives, for $M > 1$ and $y \geq 0$

$$u_1 \sim u_0 \left(\frac{2\nu}{3M^5 (M^2-1)^{3/2} cy} \right)^{1/4} e^{-\frac{c\sqrt{M^2-1}\,\left(x - y\sqrt{M^2-1}\right)^2}{\frac{8}{3}\nu M^3 y}} \tag{3.413}$$

$$u_2 \sim u_0 \operatorname{erfc} \frac{y}{2\sqrt{\frac{x\nu}{U}}} \tag{3.414}$$

The v components v_1 and v_2 can also be computed; one finds that v_1 is given by

$$v_1 \sim -u_0 \left(\frac{2\nu}{3M^5 (M^2-1)^{1/2} cy} \right)^{1/4} e^{-\frac{c\sqrt{M^2-1}\,\left(x - y\sqrt{M^2-1}\right)^2}{\frac{8}{3}\nu M^3 y}} \tag{3.415}$$

while v_2 vanishes as $\frac{1}{\sqrt{x}}$.

In particular, one notes from an examination of formulas (3.413) and (3.415) that the resultant velocity vector $\vec{q}_1$ is perpendicular to the direction of the Mach line, just as in the case of the fundamental solution (cf. §2.8). Its magnitude is

$$|\vec{q}_1| = u_0 \left(\frac{2\nu}{3M(M^2-1)^{3/2} cy} \right)^{1/4} e^{-\frac{c\sqrt{M^2-1}\,\left(x - y\sqrt{M^2-1}\right)^2}{\frac{8}{3}\nu M^3 y}} \tag{3.416}$$

The pressure field is given by

$$p_1 = -\rho_0 U u_0 \left(\frac{2\nu}{3M^5 (M^2-1)^{3/2} cy} \right)^{1/4} e^{-\frac{c\sqrt{M^2-1}\,\left(x - y\sqrt{M^2-1}\right)^2}{\frac{8}{3}\nu M^3 y}} \tag{3.417}$$

Formulas (3.413–3.417) should be compared with the discussion in §3.3. Consider the values of the flow field for a fixed y. Far downstream of the Mach line (subcharacteristic) from the leading edge, $(x = y\sqrt{M^2-1})$, the longitudinal components are small. The u-component of the velocity is given mainly by its transversal part u_2. This means that the flow field is described here by the linearized boundary-layer theory discussed in §3.3 (see in particular equation 3.32).

On the other hand, near the Mach line from the leading edge the longitudinal wave is not negligible. If u_0 is negative (which is the case of practical interest) the longitudinal part represents a compression wave. It has a maximum on the line $x = y\sqrt{M^2-1}$ and decays exponentially on either side of this line. Note that the pressure perturbation satisfies the same approximate formula $p = -\rho_0 U u_1$ as the corresponding formula for the fundamental solution (cf. 2.840). It should be remembered that none of the formulas given here are valid near the leading edge. These statements about the longitudinal waves are consistent with the discussion of the iteration procedure in §3.3.

The above analysis of the solution of the integral equations (3.412) is preliminary and will be discussed in more detail in a forthcoming report. This report will also contain a discussion of the subsonic case, which may be treated in a way similar to the supersonic case. It is also intended to investigate conditions near the leading edge (low local Reynolds numbers).

4. CONCLUDING REMARKS

In the main body of the report and in the appendices, various problems and their solutions are discussed. Now a brief evaluation of these results will be given with special reference to their relationship to the motivating problems mentioned in the Introduction.

Most of the work dealt with problems for a set of linearized equations of motion of a fluid with negligible heat conduction. It was found that the linearized waves may be split into two types, longitudinal and transversal. The nature and propagation of these waves was studied in some detail. On the basis of the approximation, the question of the relationship of the characteristics of the non-viscous equations to the solution of the equations with viscosity has also been discussed. The linearization here permits the treatment of a problem which exists for the full non-linear equations and the results obtained are significant. The results, as indicated by several examples, are the following: The propagation of a viscous wave, in certain important cases, may be regarded as the superposition of two effects. The first effect is a propagation of the non-viscous type which is determined by the characteristics of the non-viscous equations (subcharacteristics). The pertinent characteristics are the hyperbolic characteristics (Mach lines) for the longitudinal waves, and the streamlines when transversal waves are considered. The second effect propagating a viscous wave is viscous diffusion from the subcharacteristics. This diffusion is due to the parabolic nature of the equations and is very similar to the propagation of heat. The description above is always valid for transversal waves, with the rate of diffusion depending on the viscosity. However, in the case of longitudinal waves, the description is valid only asymptotically for large values of certain local Reynolds numbers.

The last remark brings to light a related problem which also exists for the non-linear equations, namely the way in which viscosity, as expressed by the Reynolds number, and compressibility, as expressed by the speed of sound or Mach number, influence the solution. It was seen that for high local Reynolds numbers the propagation of the longitudinal wave was dominated by the compressibility effect, since the hyperbolic characteristics are just those lines along which disturbances due to compressibility propagate in a non-viscous fluid. However, for low local Reynolds numbers the longitudinal wave shows only diffusion of parabolic type. The viscous effect is dominant and the effects of compressibility are secondary. The subcharacteristics and Mach number do not have any special significance. Now, in any given problem, both high and low local Reynolds numbers exist. The domain in which the solution will show, for example, the low Reynolds number behavior, depends on the actual magnitude of the physical quantities involved. Both cases, low and high Reynolds number, were obtained as special cases of a solution valid for all space and time. For longitudinal waves the solutions are given in the form of contour integrals and further analysis may indicate the nature of the flow in the intermediate range of Reynolds numbers. For the transversal waves, it was shown that compressibility has no effect at all. In this case, the solutions are obtained in closed form in terms of functions related to the solutions of the heat equation.

After a general study of pure longitudinal or transversal waves, the interaction between the two types was treated. In a linearized theory, any number of the two types of waves may be superimposed and the result is still a solution. Thus there is never any local interaction or coupling of the waves at a point in the flow. Interaction here refers only to the fact that in certain problems both types of waves are needed in order to satisfy the boundary conditions. For example, if a transversal wave is used as the first approximation, it may happen that not all boundary conditions are satisfied. A longitudinal wave may have to be added to correct this, and in this case the longitudinal wave is said to be induced by the transversal wave. The boundary-value problems actually studied were those connected with a flat plate at zero angle of attack moving through the fluid. For the flat plate, there is such an interaction. Rather complete results showing this were obtained for the plate of infinitesimal length. They are shown most clearly by the fundamental solutions which were found for the complete equations. Intuitively, the fundamental solution may be thought of as describing the flow field of a very short flat plate with a very strong retarding action (singular flat plate). It gives the complete picture, involving both types of waves, in one step.

These results supply a partial answer to the question of what part of the complete solution is given by the classical boundary-layer theory. It appears that the boundary-layer theory gives only a transversal wave which is not a good approximation to the transversal wave generated by the infinitesimal plate. The plate also generates a longitudinal wave, while according to boundary-layer theory there should be no longitudinal disturbances at all. However, the question of the validity of boundary-layer theory is only partially answered on the basis of a singular flat plate because there is no variation of Reynolds number along the plate. For that reason, the case of a semi-infinite plate was also studied, although not completely solved. Some integral equations for the complete solution were found and asymptotic or approximate results were obtained from them. Also, a rather intuitive iteration procedure involving both longitudinal and transversal waves was carried out. On the basis of these considerations, the following comments can be made about the boundary-layer theory; the boundary-layer solution describes the flow far downstream of the leading edge. It is an asymptotic solution valid when the Reynolds number based on the distance from the leading edge is high. Near the leading edge, the Reynolds number is low and the conditions are quite different. There the transversal wave is not of the boundary-layer type; e.g. the effect of the plate propagates upstream, although exponentially damped. This agrees with the idea of parabolic type diffusion of viscous waves mentioned before. In addition, "induced" longitudinal waves, which are not predicted by the boundary-layer theory, exist everywhere. They are particularly strong around the leading edge. In the supersonic case these longitudinal waves propagate along the Mach lines (sub-characteristics) far into the outer fluid. At the same time, they are diffused about those lines just as the one-dimensional waves were. The extent of the "region of the nose" where the Reynolds number is low depends on the magnitude of the physical quantities involved such as ρ, μ, U and the length of the body. When the overall Reynolds number of a body is small, the "nose type" flow might apply all over the body. This may occur, for example, when ρ is small. Of course, a further study of the integral equations is needed in order to make the statements above more precise and in order to describe the entire flow field at once. Similar problems for a finite flat plate should also be studied.

A special appendix is devoted to the derivation of the fundamental solutions for various cases. Such solutions are of basic importance for a treatment of linear equations. Actually, they were used in the treatment of most of the problems mentioned above. Many results are to be expected from a further study of the fundamental solutions. As indicated in Appendix C, they may even be used in solving non-linear problems by iteration methods.

In summary, this report contains the basis of a technique for dealing with the linearized equations and many examples of the use of this technique. It seems capable of further development in particular by further analysis and use of the fundamental solutions and by study of the various integral equations derived for boundary-value problems. As mentioned in the Introduction, much of the material on the linearized theory is not new. However, older investigations of the viscosity have been rather neglected in aerodyanmics in favor of the boundary-layer theory.

It is important to remember that all of the comments made above are based wholly on a linearized theory. The restriction to linearized flow is rather severe. For example, in the case of the flat plate mentioned above, the linearization means that, to the first order, the Mach number is the same everywhere in the fluid. The Reynolds number, however, shows a wide variation. Thus, although the joint effect of Reynolds number and Mach number was studied, no provision was made for appreciable changes in the local Mach number. In a supersonic flow, such changes must occur if any bodies are present in the fluid.

Some preliminary work of an approximate nature has been done in this report to find the general behavior of the non-linear effects. Non-stationary longitudinal and transversal waves in one dimension were considered. In the longitudinal case the non-linear effect which is accounted for is the tendency of a wave front to steepen due to a non-linear acceleration term. The results indicate that the idea of a longitudinal wave propagating along the subcharacteristics with a superimposed viscous dispersion is still valid. However, the non-linearity counteracts this dispersion to some extent so that the wave does not necessarily attenuate but can maintain a finite amplitude. In the two-dimensional stationary case the non-linearity would also permit curved subcharacteristics and it is to be expected that the longitudinal waves diffuse about these. In the case of the transversal wave a different non-linear effect was studied. It was shown how a transversal wave, because of its dissipation, generates a longitudinal wave. The solution was only approximate but could be made the basis of an iteration procedure. This type of effect is very important close to the plate where the actual velocity gradients of the transversal wave are large. Judging by the results of the linearized boundary-layer theory, this type of effect may be an important part of a theory of flow near the plate, more exact than boundary-layer theory.

In a discussion of such dissipative effects, the heat conduction becomes important. Preliminary work indicates that temperature waves also exist which may affect the nature of the solution considerably. However, in all the linearized work and in the weak non-linear longitudinal waves the effect of the heat conduction is qualitatively unimportant. The heat conduction can be included at the cost of complicating all the formulas without changing their character. The heat conduction would also be important in fairly strong shock waves where the dissipation must be considered.

There is one potentially important method which has not been utilized in the present report. This is the perturbation method applied to the parameters of the differential equations (see §1.7). However, in the light of the results of this report some significant problems to be treated by this method can be mentioned. The outstanding one is the significance of the subcharacteristics in the singular perturbation problem $\mu \to 0$. This could be carried out first for a linearized case in order to see just how much a singular perturbation problem can yield, and then it could be extended to the non-linear case. The other interesting perturbation problems concern the behavior of the flow as $\mu \to \infty$, $c \to \infty$ or $\rho \to 0$.

Thus, general problems have been studied within the linearized theory. In addition, some attempts were made at a non-linear theory of pure longitudinal or transversal waves. In each of these two cases, special non-linear effects occur in addition to the features discovered by a linearized treatment. Many of the problems mentioned in the Introduction concerned an interaction of transversal and longitudinal wave. The linearized theory developed in this report may describe some aspects of these problems. There must also occur typical non-linear effects which have not manifested themselves in the simple cases of pure waves and which have no counterpart in linearized theory. However, a discussion of the exact problem is so difficult that it was felt necessary to clear up the simplest problems first as a preparation for future work. Thus, although we are still rather far from a mathematical description of a real object moving in a real fluid, we hope that some steps in the right direction have been taken.

APPENDIX A

Derivation of Results by Laplace Transformation; Consideration of a Certain Integral

In this appendix a detailed presentation will be given of the application of the method of Laplace transformation to specific problems for the equation of propagation of a one-dimensional viscous perturbation (cf. §2.2)

$$\frac{4}{3}\nu u_{xxt} + c^2 u_{xx} - u_{tt} = 0 \tag{A1}$$

The problems consider the spread of a step or impulse signal in gas initially at rest. An application of the standard methods yields the solution immediately in the form of a contour integral. The main work is analysis of this contour integral which yields certain approximate results. Although only the simplest problems are considered in detail it will be clear how analogous results could be obtained for arbitrary initial conditions and boundary values. The integrals which occur in the solution of the specific problems are special cases of a more general integral, important in the general theory of the linearized viscous equations, which will also be discussed.

If the dimensionless variables $\frac{c^2 t}{4/3\,\nu}$, $\frac{cx}{4/3\,\nu}$ are replaced by t, x the equation may be written

$$u_{xxt} + u_{xx} - u_{tt} = 0 \tag{A1'}$$

A solution must be obtained in the region $\begin{matrix} x > 0 \\ t > 0 \end{matrix}$, satisfying the following conditions:

Initial conditions: gas at rest $u(x,0) = 0$, $u_t(x,0) = 0$ (A2) and boundary conditions:

$$u(0,t) = 0 \qquad t<0$$

Either step signal $\quad u(0,t) = u_o \qquad t>0$ **(A3a)**

Or impulse signal $\quad u(0,t) = 0 \qquad t<0$

$$= \frac{u_o}{\delta} \quad 0 \leq t \leq \delta \quad \text{where} \quad \delta \to 0 \tag{A3b}$$

$$= 0 \qquad t > \delta$$

and damping out of disturbances at $x = \infty$

Writing $\bar{u}(x;\sigma)$ for the Laplace transform of $u(x;t)$, we have

$$\bar{u}(x;\sigma) = \int_0^\infty e^{-\sigma t} u(x,t)\, dt \tag{A4}$$

and taking account of the zero initial condition (A2) we obtain from (A1') the following subsidiary equation for $\bar{u}$

$$(1+\sigma)\,\bar{u}_{xx} - \sigma^2 \bar{u} = 0 \tag{A5}$$

The Laplace transform $\bar{u}(x;\sigma)$ is obtained as the solution of (A5) under the transformed boundary condition:

either step signal $\quad \bar{u}(0;\sigma) = \int_0^\infty e^{-\sigma t} u_o\, dt = \frac{u_o}{\sigma}$ **(A6a)**

or impulse signal $\quad \bar{u}(0;\sigma) = \int_0^\delta e^{-\sigma t} u_o\, dt = \frac{u_o}{\delta\sigma}\left(1 - e^{-\sigma\delta}\right) = u_o$ **(A6b)**

as $\delta \to 0$

and damping of $\bar{u}(x;\sigma)$ at $x = \infty$

Thus

$$\bar{u}(x;\sigma) = u_o\, e^{-x\frac{\sigma}{\sqrt{1+\sigma}}}\, \frac{1}{\sigma} \qquad \text{step signal} \tag{A7a}$$

$$= u_o\, e^{-x\frac{\sigma}{\sqrt{1+\sigma}}} \qquad \text{impulse signal} \tag{A7b}$$

Thus the problem is reduced to finding the inverse Laplace transform and this may be accomplished formally by the use of the complex inversion formula

$$u(x,t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{\sigma t} \, \bar{u}\,(x;\sigma)\, d\sigma \tag{A8}$$

where the contour lies to the right of all the singularities of $\bar{u}(x;\sigma)$ in the complex σ -plane. For a discussion of the validity of (A8) and the conditions assumed to be satisfied by $\bar{u}\,(x;\sigma)$ see Ref. 2. Hence the solutions can be expressed as contour integrals

$$u(x,t) = \frac{u_o}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{\sigma t - \frac{\sigma}{\sqrt{1+\sigma}}x} \frac{d\sigma}{\sigma} \quad \text{for the step signal} \tag{A9a}$$

$$= \frac{u_o}{2\pi i} \int_{C-i\infty}^{C+i\infty} e^{\sigma t - \frac{\sigma}{\sqrt{1+\sigma}}x} d\sigma \quad \text{for the impulse signal} \tag{A9b}$$

The singularities of the integrand are a branch point and an essential singularity at $\sigma = -1$ and in the first case a pole at $\sigma = 0$, so that in general one must restrict $|\arg(1+\sigma)| \leq \pi$. It can be verified in the usual way that (A9a), (A9b) do satisfy the differential equation and the proper boundary conditions and represent a unique solution. It is possible to derive some approximate formulae for $u(x,t)$ from (A9a), (A9b). Results useful for large values of t are

$$u(x,t) = \frac{u_o}{2}\left(1 - \operatorname{erf}\frac{x-t}{\sqrt{2t}}\right) + \mathcal{E}_0 \quad \text{for step signal} \tag{A10a}$$

$$= \frac{u_o}{2\sqrt{2\pi t}}\, e^{-\frac{(x-t)^2}{2t}} + \mathcal{E}_1 \quad \text{for impulse signal} \tag{A10b}$$

where

$$|\mathcal{E}_0| < \frac{1}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

$$|\mathcal{E}_1| < \frac{1}{t} + o\left(\frac{1}{t}\right)$$

A derivation of these results follows.

The integrals (A9a), (A9b) can be considered special cases of a slightly more general integral

$$\mathcal{J}(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\sigma t - \frac{\sigma}{\sqrt{1+\sigma}}x}\, \sigma^{a-1}\, F(\sigma)\, d\sigma \qquad \textbf{(A11)}$$

$$a = 0, 1, 2, \ldots \quad (t>0,\ x>0)$$

which will be discussed here. $F(\sigma)$ is a function with suitable properties of boundedness etc. The contour then can be taken to be the imaginary axis, indented to the right of the origin if necessary ($a=0$, only) (see Fig. A1). The first step is to show that the contour AOB can be deformed into a contour like C' since the contribution along the large semi-circles BC and DA will vanish as $R \to \infty$. Let $\mathcal{J}_1 = \int_B^C$ and let $\sigma = -1 + Re^{i\theta}$ on BC. Then

$$\mathcal{J}_1 = \frac{1}{2\pi i}\int_{\cos^{-1}\frac{1}{R}}^{\pi} e^{(-1+Re^{i\theta})t - x\frac{(-1+Re^{i\theta})}{R^{1/2}e^{i\theta/2}}} (-1+Re^{i\theta})^{a-1} \cdot F(Re^{i\theta}-1)Re^{i\theta}\, i\, d\theta \qquad \textbf{(A12)}$$

and let $\mathcal{J}_{11} = \int_{\cos^{-1}\frac{1}{R}}^{\Lambda}$ and $\mathcal{J}_{12} = \int_{\Lambda}^{\pi}$ so that $\mathcal{J}_1 = \mathcal{J}_{11} + \mathcal{J}_{12}$ **(A13)**

and Λ is arbitrary

Now

$$|\mathcal{J}_{11}| < \frac{R(R-1)^{a-1}}{2\pi}\, \mathcal{M}(R) \int_{\cos^{-1}\frac{1}{R}}^{\Lambda} e^{(R\cos\theta - 1)t - x\left(R^{\frac{1}{2}} - R^{-\frac{1}{2}}\right)\cos\frac{\theta}{2}}\, d\theta$$

where

$$|F(Re^{i\theta}-1)| < \mathcal{M}(R)$$

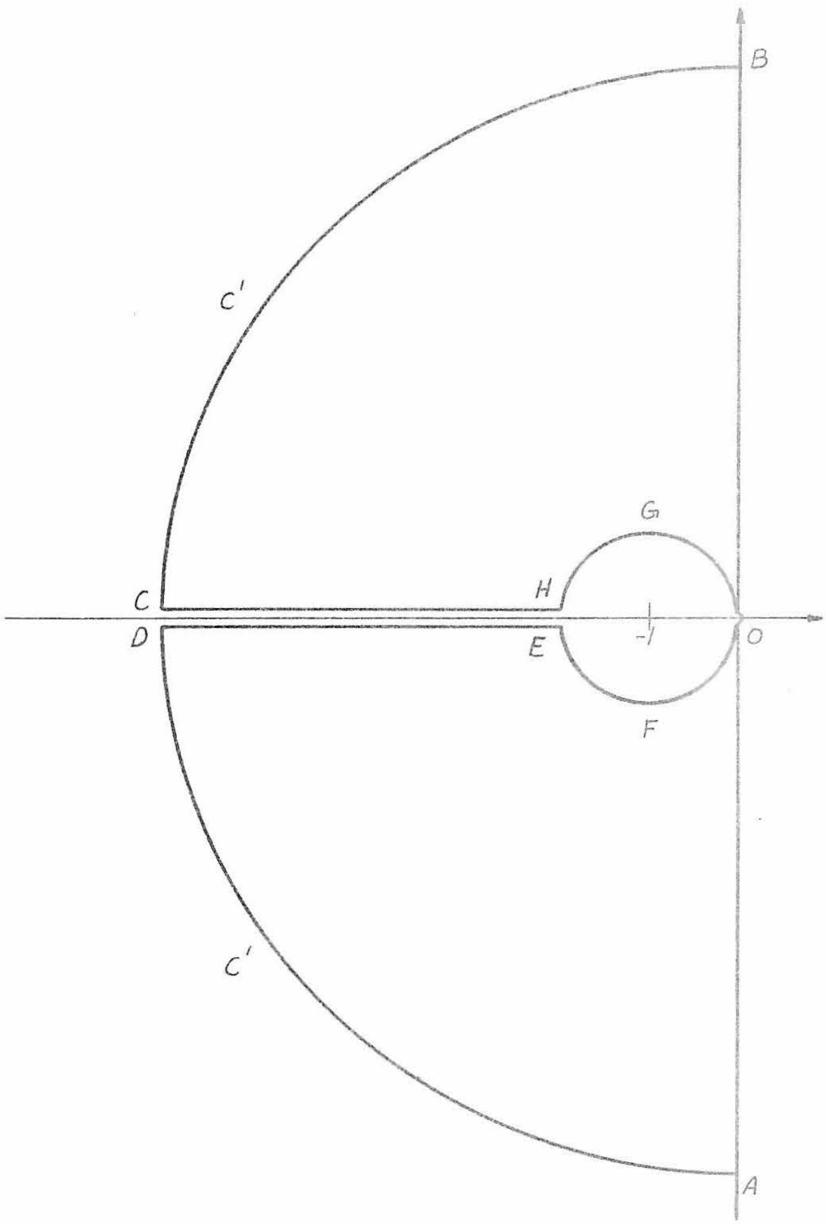

Figure A1

for

$$\cos^{-1}\frac{1}{R} \le \theta \le \Lambda \le \pi$$

Thus

$$|\mathcal{J}_{11}| < \frac{R(R-1)^{a-1}}{2\pi}\, m(R)\left(\Lambda - \cos^{-1}\tfrac{1}{R}\right) e^{1-t} \left\{ e^{-x\left(R^{\frac{1}{2}}-R^{-\frac{1}{2}}\right)\cos\frac{\Lambda}{2}} \right\}$$

so that $|\mathcal{J}_{11}| \longrightarrow 0$ as $R \to \infty$, if $m(R)$ does not approach ∞ faster than $e^{xR^{\frac{1}{2}}}$. Next

$$|\mathcal{J}_{12}| < \frac{R(R-1)^{a-1}\, n(R)}{2\pi} \int_{\Lambda}^{\pi} e^{(R\cos\theta - 1)t - x\left(R^{\frac{1}{2}}-R^{-\frac{1}{2}}\right)\cos\frac{\theta}{2}}\, d\theta$$

$$< \frac{R(R-1)^{a-1}}{2\pi}\, n(R)(\pi - \Lambda)\, e^{(R\cos\Lambda - 1.)t}$$

so that $|\mathcal{J}_{12}| \longrightarrow 0$ as $R \to \infty$ since $\cos\Lambda < 0$, and it is assumed that

$n(R) = \max\left|F(Re^{i\theta}-1)\right| \quad \Lambda \le \theta < \pi$, does not approach ∞ more rapidly than $e^{Rt\cos\Lambda}$. It follows from (13) that $|\mathcal{J}_{11}| \to 0$, $R \to \infty$, and of course the same arguments can be used for the integral taken along DA. Hence, the integral along a path starting at $\sigma = -\infty$ below the cut on the negative real axis, circling the origin in a counter-clockwise direction, and extending to $\sigma = -\infty$ above the negative real axis, is equal to the original integral $\mathcal{J}(x,t)$. The next step is to choose a contour of this type which aids approximation. It is somewhat desirable to have $\frac{\sigma}{\sqrt{1+\sigma}}$ a purely imaginary quantity for purposes of estimation. Hence the negative real axis on both sides of the cut from $-\infty$ to -2 and a circle of unit radius about $\sigma = -1$, indented at the origin if necessary, is taken as the path of integration (path $DEFOGHC$). For large values of t most of the contribution comes from small $|\sigma|$, or from the integral along the path near O. The size of the integrals on the other parts of the path can be estimated. An expression for $\mathcal{I}$ in terms of known integrals and an error term will now be found.

Let $\mathcal{I}_2 = \int_O^E$ and let $\sigma = 1 + re^{-i\pi}$ on DE

Then from (11)

$$\mathcal{I}_2 = \frac{(-)^{a-1} e^{-t}}{2\pi i} \int_1^\infty e^{-rt} e^{ix \frac{(1+r)}{\sqrt{r}}} (1+r)^{a-1} F_1(r)\, dr$$

where

$$F_1(r) = F\left(-1 + re^{-i\pi}\right)$$

and we assume

$$\left|F_1(r)\right| < K_1 , \qquad 1 \leq r \leq \infty$$

Then, either $a = 0$ and

$$\left|\mathcal{I}_2\right| < \frac{K_1 e^{-t}}{2\pi} \int_1^\infty e^{-rt} \frac{dr}{1+r} < \frac{K e^{-t}}{2\pi} \int_1^\infty e^{-rt}\, dr$$

$$\left|\mathcal{I}_2\right| < \frac{K_1 e^{-2t}}{2\pi t} \tag{A14}$$

or, $a > 0$ and

$$\left|\mathcal{I}_2\right| < \frac{K_1 e^{-t}}{2\pi} \int_1^\infty e^{-rt} (1+r)^{\overline{a-1}}\, dr < \frac{K_1 e^{-t}}{2\pi} \int_1^\infty e^{-rt} (2r)^{a-1}\, dr$$

$$\left|\mathcal{I}_2\right| < \frac{K_1 e^{-t}}{2\pi}\, 2^{a-1} \int_0^\infty e^{-rt} r^{a-1}\, dr$$

so that

$$\left|\mathcal{I}_2\right| < \frac{2^{a-2} K_1}{\pi} \frac{\Gamma(a)}{t^a} e^{-t} \tag{A15}$$

by virtue of the well known formula

$$\int_0^\infty e^{-rt} r^m\, dr = \frac{\Gamma(m+1)}{t^{m+1}} , \qquad m > -1 \tag{A16}$$

In either case $|\mathcal{J}_2|$ damps out exponentially in time and of course the same result can be deduced for the integral along HC.

Next the integrals on the unit circle from Λ' to π, and $-\Lambda'$ to $-\pi$ will be estimated.

Let $\mathcal{J}_3 = \int_G^H$ and let $\sigma = -1+e^{i\theta}$ on GH

Then, from (A11)

$$\mathcal{J}_3 = \frac{1}{2\pi}\int_{\Lambda'}^{\pi} e^{(-1+e^{i\theta})t}\, e^{-\frac{(e^{i\theta}-1)x}{e^{i\theta/2}}}\,(-1+e^{i\theta})^{a-i} F_2(\theta)e^{i\theta}\,d\theta \quad \text{where} \quad F_2(\theta)=F(e^{i\theta}-1)$$

$$= \frac{(2i)^{a-1}}{2\pi}\int_{\Lambda'}^{\pi} e^{(-1+e^{i\theta})t}\, e^{-2i\sin\frac{\theta}{2}x}\left(\sin\frac{\theta}{2}\right)^{a-1} e^{i\theta\left(\frac{a+1}{2}\right)} F_2(\theta)\,d\theta$$

$$|\mathcal{J}_3| < \frac{2^{a-2}}{\pi} K_2 \int_{\Lambda'}^{\pi} e^{(\cos\theta-1)t}\left(\sin\frac{\theta}{2}\right)^{a-1} d\theta \quad \text{where} \quad |F_2(\theta)|<K_2,\ \Lambda'\leq\theta\leq\pi$$

$$|\mathcal{J}_3| < \frac{2^{a-2}}{\pi} K_2 \frac{e^{-(1-\cos\Lambda')t}}{\sin\frac{\Lambda'}{2}} \quad \text{and} \quad \Lambda'>0 \tag{A17}$$

Thus $\mathcal{J}_3$ also vanishes exponentially for large t and a similar result can be obtained for the integral on EF. Thus we must evaluate

$$\mathcal{J}_4 = \frac{1}{2\pi}\int_{-\Lambda'}^{\Lambda'} e^{(\cos\theta-1)t+ti\sin\theta-2i\pi\sin\frac{\theta}{2}}(e^{i\theta}-1)^{a-1}e^{i\theta}F(e^{i\theta}-1)\,d\theta$$

First we can transform the integral by writing

$$\omega=\sin\frac{\theta}{2}, \quad d\theta=\frac{d\omega}{\frac{1}{2}\sqrt{1-\omega^2}}, \quad \omega_1=\sin\frac{\Lambda'}{2}$$

and let

$$\psi(\omega)=\left(e^{i\frac{\theta}{2}}\right)^{a+1}\frac{G(\omega)}{\sqrt{1-\omega^2}}=\left(\sqrt{1-\omega^2}+i\omega\right)^{a+1}\frac{G(\omega)}{\sqrt{1-\omega^2}}$$

where

$$G(\omega)=F\left(-2\omega^2+2i\omega\sqrt{1-\omega^2}\right)$$

Then

$$\mathcal{J}_4=\frac{(2i)^{a-1}}{\pi}\int_{-\omega_1}^{\omega_1} e^{-2t\omega^2+2i\omega\left(\sqrt{1-\omega^2}\,t-x\right)}\,\omega^{a-1}\,\psi(\omega)\,d\omega+\mathcal{J}_{41}$$

Now we can approximate part of the integral by its value for small ω .

We may write $\psi(\omega) = \psi(0) + \omega\,\psi'(A\omega)$, $0<A<1$, by a mean value theorem. Thus

$$\mathcal{J}_4 = \frac{(2i)^{a-1} F(0)}{\pi} \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2 + 2i\omega(\sqrt{1-\omega^2}\,t - x)} \omega^{a-1}\, d\omega + \mathcal{J}_{41}$$

for

$$\psi(0) = G(0) = F(0) \text{ and } \quad \mathcal{J}_{41} = \frac{(2i)^{a-1}}{\pi} \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2 + 2i\omega(\sqrt{1-\omega^2}\,t - x)} \omega^a \psi'(A\omega)\, d\omega$$

But

$$\psi'(\omega) = i(a+1)\left(\sqrt{1-\omega^2} + i\omega\right)^{a+1} \frac{G(\omega)}{1-\omega^2} + \left(\sqrt{1-\omega^2}+i\omega\right)^{a+1} \left\{ \frac{G'(\omega)}{\sqrt{1-\omega^2}} + \frac{\omega G(\omega)}{(1-\omega^2)^{3/2}} \right\}$$

so that the remainder integral $\mathcal{J}_{41}$ can now be estimated as

$$\left|\mathcal{J}_{41}\right| < \frac{2^{a-1}}{\pi} \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2} \left|\omega^a\right| \left\{ \frac{|G(A\omega)|}{1-\omega^2}(a+1) + \frac{|G'(A\omega)|}{\sqrt{1-\omega_1^2}} + \frac{|A\omega\, G(A\omega)|}{\left(\sqrt{1-\omega_1^2}\right)^3} \right\} d\omega$$

Now, by using

$$\int_{-\omega_1}^{\omega_1} e^{-2t\omega^2} \left|\omega^m\right| d\omega < \int_0^\infty e^{2tZ} Z^{\frac{m-1}{2}}\, dz = \frac{\Gamma\left(\frac{m+1}{2}\right)}{(2t)^{\frac{m+1}{2}}}$$

and assuming

$$\left|G(A\omega)\right| < M_1$$
$$\left|G'(A\omega)\right| < M_2$$

we see

$$\left|\mathcal{J}_{41}\right| < \frac{2^{a-1}}{\pi} \left\{ \frac{(a+1)M_1 + M_2\sqrt{1-\omega_1^2}}{1-\omega_1^2} \, \frac{\Gamma\left(\frac{a+1}{2}\right)}{(2t)^{\frac{a+1}{2}}} + \frac{M_1}{(1-\omega_1^2)^{\frac{3}{2}}} \, \frac{\Gamma\left(\frac{a+2}{2}\right)}{(2t)^{\frac{a+2}{2}}} \right\} \qquad \text{(A18)}$$

Next, a slightly more delicate approximation is needed to see the effect of small ω in the exponents. We may write

$$\mathcal{J}_4 = \frac{(2i)^{a-1}}{\pi} F(0) \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2 + 2i\omega(t-x)} g(\omega)\, \omega^{a-1}\, d\omega + \mathcal{J}_{41}$$

where

$$g(\omega) = e^{2i\omega t\left(\sqrt{1-\omega^2}-1\right)}$$

and use $\quad g(\omega) = g(0) + \omega g'(B\omega) \,, \quad 0 < B < 1$

$$g'(\omega) = -4it \frac{e^{2i\omega t\left(\sqrt{1-\omega^2}-1\right)}\,\omega^2}{\sqrt{1-\omega^2}}$$

so that

$$\mathcal{J}_4 = \frac{(2i)^{a-1} F(0)}{\pi} \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2 + 2i\omega(t-x)}\, \omega^{a-1}\, d\omega + \mathcal{J}_{42} + \mathcal{J}_{41}$$

where

$$\mathcal{J}_{42} = -\frac{(2i)^{a-1} F(0)}{\pi}\, 4it\, B^2 \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2 + 2i\omega(t-x)} \frac{e^{2iB\omega t\left(\sqrt{1-B^2\omega^2}-1\right)}}{\sqrt{1-B^2\omega^2}}\, \omega^{a+2}\, d\omega$$

This may be estimated by

$$\left|\mathcal{J}_{42}\right| < \frac{2^{a+1}}{\pi} \left|F(0)\right| \frac{t}{\sqrt{1-\omega_1^2}} \int_{-\omega_1}^{\omega_1} e^{-2t\omega^2} \left|\omega^{a+2}\right| d\omega$$

so that

$$\left|\mathcal{J}_{42}\right| < 2^{a+1} \frac{\left|F(0)\right|}{\pi} \frac{1}{\sqrt{1-\omega_1^2}} \frac{\Gamma\left(\frac{a+3}{2}\right)}{2^{\frac{a+3}{2}}\, t^{\frac{a+1}{2}}} \qquad \textbf{(A19)}$$

Finally $\mathcal{J}_4$ can be related to known integrals by extending the range of integration to ∞, and the error introduced may be estimated. Then two cases must be distinguished.

Case 1. a even:

$$\mathcal{J}_4 = \frac{2^a}{\pi} (-)^{\frac{a}{2}} F(0) \int_0^\infty e^{-2t\omega^2} \sin 2\omega(t-x)\, \omega^{a-1}\, d\omega + \mathcal{J}_{42+41} + \mathcal{J}_{43}$$

where

$$\mathcal{J}_{43} = \frac{2^a}{\pi} (-)^{\frac{a}{2}} F(0) \int_{\omega_1}^\infty e^{-2t\omega^2} \sin 2\omega(t-x)\, \omega^{a-1}\, d\omega$$

Now

$$\left|\mathcal{J}_{43}\right| \leq \frac{2^a}{\pi}\left|F(0)\right|\int_{\omega_1}^{\infty}\left(e^{-t\omega^2}\omega^{\frac{1}{2}}\right)\left(e^{-t\omega^2}\omega^{a-\frac{2}{3}}\right)d\omega$$

$$\leq \frac{2^a}{\pi}\left|F(0)\right|\sqrt{\int_{\omega_1}^{\infty} e^{-2t\omega^2}\omega\, d\omega \int_{\omega_1}^{\infty} e^{-2t\omega'^2}\omega'^{\,2a-3}\, d\omega'}$$

$$\leq \frac{2^a}{\pi}\left\{\frac{e^{-t\omega_1^2}}{\sqrt{2t}}\right\}\left\{\frac{1}{\omega_1^{3-2a}}\ \frac{1}{2}\sqrt{\frac{\pi}{2t}}\right\}\left|F(0)\right| \text{ for } \quad a=0,1 \qquad \textbf{(A20a)}$$

or

$$\left|\mathcal{J}_{43}\right| \leq \frac{2^a}{\pi}\left|F(0)\right|\left\{\frac{e^{-t\omega_1^2}}{\sqrt{2t}}\right\}\left\{\frac{1}{2}\,\frac{\Gamma(a-1)}{(2t)^{a-1}}\right\}^{\frac{1}{2}}, \quad a=2,3,4\ldots \qquad \textbf{(A20)}$$

so that $\left|\mathcal{J}_{43}\right|$ vanishes exponentially for large t.

Case 2. a odd:

$$\mathcal{J}_4 = \frac{2^a}{\pi}(-)^{\frac{a-1}{2}}F(0)\int_0^{\infty} e^{-2t\omega^2}\cos 2\omega(t-x)\,\omega^{a-1}\,d\omega + \mathcal{J}_{42+41} + \mathcal{J}_{44}$$

where

$$\mathcal{J}_{44} = \frac{2^a}{\pi}(-)^{\frac{a-1}{2}}F(0)\int_{\omega_1}^{\infty} e^{-2t\omega^2}\cos 2\omega(t-x)\,\omega^{a-1}\,d\omega$$

and has the same estimate as $\mathcal{J}_{43}$ above.

The infinite integrals occurring in $\mathcal{J}_4$ can be related to known integrals by the following formulas. If we consider

$$\mathcal{F}(b,t) = \int_0^{\infty} e^{-2t\omega^2}\cos 2\omega b\, d\omega = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2t}}\, e^{-\frac{b^2}{2t}}$$

then

$$\int_0^b \mathcal{F}(\alpha,t)\,d\alpha \equiv \frac{1}{2}\int_0^{\infty} e^{-2t\omega^2}\frac{\sin 2\omega b}{\omega}\,d\omega = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2t}}\int_0^b e^{-\frac{\alpha^2}{2t}}\,d\alpha$$

$$= \frac{\sqrt{\pi}}{4}\int_0^{\frac{b}{\sqrt{2t}}} e^{-\sigma^2}\,d\sigma$$

$$= \frac{\pi}{4}\,\mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right)$$

Then, defining

$$E_o(b,t) \equiv \frac{1}{\pi}\int_0^{\infty} e^{-2t\omega^2}\,\frac{\sin 2\omega b}{\omega}\,d\omega = \frac{1}{2}\,\mathrm{erf}\left(\frac{b}{\sqrt{2t}}\right) \tag{A21}$$

and more generally, defining

$$E_a(b,t) = \frac{2^{a-1}}{\pi}(i)^{a-1}\int_{-\infty}^{\infty} e^{-2t\omega^2}\,e^{i\omega b}\,\omega^{a-1}\,d\omega \tag{A22}$$

we notice that

$$E_a(b,t) = \frac{\partial^a}{\partial b^a}\,E_o(b,t) \tag{A23}$$

Thus, the original integral may be given the following evaluation

$$\mathcal{J}_a(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma t - \frac{\sigma x}{\sqrt{1+\sigma}}}\,\sigma^{a-1}\,F(\sigma)\,d\sigma = F(0)\,E_a(t-x,t) + \mathcal{E}_a + \left[\frac{1}{2}\right] \tag{A24}$$

for a residue of $\frac{1}{2}$ has to be taken into account at the origin in the case when $a=0$. $\mathcal{E}_a$ denotes the integrals which have been estimated. By considering (A20), (A19), (A18), (A17), (A15) and (A14), and the number of times which similar integrals occur, an upper bound for the error $\mathcal{E}_a$ can be found. If only the largest terms are written out, and arbitrarily, $\beta = \frac{\pi}{2}$, $\omega_1 = \frac{1}{\sqrt{2}}$

$$\left|\mathcal{E}_a\right| < \frac{2^{\frac{a+1}{2}}}{\sqrt{3}\,\pi}\,|F(0)|\,\frac{\Gamma\left(\frac{a+3}{2}\right)}{t^{\frac{a+1}{2}}} + \frac{2^{\frac{a+1}{2}}}{\sqrt{3}\,\pi}\,\frac{\Gamma\left(\frac{a+1}{2}\right)}{t^{\frac{a+1}{2}}}\left\{(a+1)\,M_1 + \frac{M_2}{\sqrt{2}}\right\} + o\left(\frac{1}{t^{\frac{a+1}{2}}}\right) \tag{A25}$$

where $o\left(\frac{1}{t^{\frac{a+1}{2}}}\right)$ indicates terms which vanish more rapidly than $\frac{1}{t^{\frac{a+1}{2}}}$ as $t \rightarrow 0$.

In the special case of $F=1,\ M_1=1,\ M_2=0$. Thus

if $a=0$

$$\left|\mathcal{E}_o\right| < \sqrt{\frac{2}{3\pi t}} + o\left(\frac{1}{\sqrt{t}}\right) \tag{A26}$$

or if $a=1$ $$|\mathcal{E}_1| < \frac{4}{\sqrt{3}\,\pi t} + O\left(\frac{1}{t}\right) \tag{A27}$$

Also

$$E_1(b,t) = \frac{\partial}{\partial b}\left\{ \frac{1}{2}\,\frac{2}{\sqrt{\pi}} \int_0^{\frac{b}{\sqrt{2t}}} e^{-\sigma^2}\, d\sigma \right\} = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{b^2}{2t}} \tag{A28}$$

Taking into consideration the estimates (A26), (A27) and the integrals (A21), (A24), and (A28) the previous results (A10a) and (A10b) are deduced, for

$$\begin{aligned} \mathcal{A}_0(x,t) &= \frac{1}{2}\left\{1 - E_0(x-t,t)\right\} + \mathcal{E}_0 \\ &= \frac{1}{2}\left\{1 - \operatorname{erf}\left(\frac{x-t}{\sqrt{2t}}\right)\right\} + \mathcal{E}_0 \end{aligned} \tag{A29a}$$

$$\mathcal{A}_1(x,t) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-t)^2}{2t}} + \mathcal{E}_1 \tag{A29b}$$

It is also desirable to determine the behavior of the solution for small values of t. At first let us consider only the integral for the step signal (A9a). A transformation valid for all $t>0$, which puts the integral into a form more suitable for estimation is $\alpha = \sigma t$. Then (A9a) becomes

$$u(x,t) = \frac{u_0}{2\pi i}\int_G e^{\alpha - \frac{\frac{x}{t}\alpha}{\sqrt{1+\frac{\alpha}{t}}}}\,\frac{d\alpha}{\alpha} \tag{A9a'}$$

where G denotes a path along the imaginary axis from $-i\infty$ to $+i\infty$, indented to the right of the pole at the origin. As before, the plane is slit, but now from $-t$ to $-\infty$ on the real axis. If we regard $\eta = \frac{x}{\sqrt{t}}$, $\tau = t$ as new variables

$$u(\eta,t) = \frac{u_0}{2\pi i}\int_G e^{\alpha - \frac{\eta\alpha}{\sqrt{t+\alpha}}}\,\frac{d\alpha}{\alpha} \tag{A30}$$

We can now obtain information about the solution near the origin by regarding η as fixed and studying (A30) for small t. By the mean value theorem

$$e^{\frac{-\eta\alpha}{\sqrt{\tau+\alpha}}} = e^{-\eta\sqrt{\alpha}} + \frac{\tau\eta\alpha}{2\left[A\tau+\alpha\right]^{3/2}}\, e^{\frac{-\eta\alpha}{\sqrt{A\tau+\alpha}}} \qquad 0<A<1 \tag{A31}$$

Substituting in (A30) we obtain,

$$u(\eta,\tau) = \frac{u_0}{2\pi i}\left[\int_G e^{\alpha-\eta\sqrt{\alpha}}\frac{d\alpha}{\alpha} + \frac{\tau\eta}{2}\int_G e^{\alpha-\frac{\eta\alpha}{\sqrt{A\tau+\alpha}}}\frac{d\alpha}{(A\tau+\alpha)^{3/2}}\right] \tag{A32}$$

The first integral gives the behavior of the solution for small τ, while the magnitude of the second can be estimated. In order to obtain the estimate we deform the path of integration (G) of the second integral so that it extends from $-i\infty$ to $-i$ on the imaginary axis, circles the origin to the right on the unit circle, and extends from $+i$ to $+i\infty$ on the imaginary axis. Then putting $\alpha = e^{\pm\frac{\pi i}{2}}\beta$ along the imaginary axis, define

$$\mathcal{J} = \int_G e^{\alpha-\frac{\eta\sqrt{\alpha}}{\sqrt{A\tau+\alpha}}}\frac{d\alpha}{(A\tau+\alpha)^{3/2}} = \int_\infty^1 e^{-i\beta-\frac{e^{\frac{-\pi i}{4}}\eta\sqrt{\beta}}{\sqrt{A\tau-i\beta}}}\frac{(-i)\,d\beta}{(A\tau-i\beta)^{3/2}} + \int_1^\infty e^{i\beta-\frac{\eta\sqrt{\beta}\,e^{\frac{\pi i}{4}}}{\sqrt{A\tau+i\beta}}}\frac{i\,d\beta}{(A\tau+i\beta)^{3/2}} + \mathcal{J}_2$$

where $\mathcal{J}_2$ denotes the integral on the unit circle. Thus if $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$

$$\mathcal{J}_1 = i\int_1^\infty e^{-i\beta-\frac{\eta\sqrt{\beta}}{(A^2\tau^2+\beta^2)^{1/4}}e^{i\left(\frac{\delta}{2}-\frac{\pi}{4}\right)}}\frac{d\beta}{(A^2\tau^2+\beta^2)^{3/4}}\,e^{\frac{3i\delta}{4}} + i\int_1^\infty e^{i\beta-\frac{\eta\sqrt{\beta}}{(A^2\tau^2+\beta^2)^{1/4}}e^{+i\left(\frac{\pi}{2}-\frac{\delta}{4}\right)}}\frac{d\beta}{(A^2\tau^2+\beta^2)^{3/4}}\,e^{\frac{3i\delta}{4}}$$

where

$$A\tau + i\beta = \sqrt{A^2\tau^2+\beta^2}\; e^{i\delta}$$

$$|\mathcal{J}_1| \le 2\int_1^\infty e^{-\frac{\eta\sqrt{\beta}\cos\left(\frac{\delta}{2}-\frac{\pi}{4}\right)}{(A^2\tau^2+\beta^2)^{1/4}}}\frac{d\beta}{(A^2\tau^2+\beta^2)^{3/4}}$$

so that

$$|\mathcal{J}_1| \leq 2\int_1^{\infty} \frac{d\beta}{\beta^{3/2}} = 4 \tag{A33a}$$

Next, on the unit circle, let $\alpha = e^{i\theta}$. Then

$$\mathcal{J}_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{e^{i\theta} - \frac{\eta e^{\frac{i\theta}{2}}}{\sqrt{A\tau + e^{i\theta}}}} \frac{e^{i\theta}\, i d\theta}{(A\tau + e^{i\theta})^{3/2}}$$

If

$$A\tau + e^{i\theta} = \sqrt{A^2\tau^2 + 2A\tau\cos\theta + 1}\; e^{i\delta}, \quad \text{then} \quad \delta = \tan^{-1}\frac{\sin\theta}{A\tau + \cos\theta}$$

and $|\delta| < |\theta|$ in the range considered. Now

$$\mathcal{J}_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{e^{i\theta} - \frac{\eta e^{\frac{i\theta}{2} - i\frac{\delta}{2}}}{(A^2\tau^2 + 2A\tau\cos\theta + 1)^{1/4}}} \frac{e^{i\theta - 3i\frac{\delta}{4}}}{(A^2\tau^2 + 2A\tau\cos\theta + 1)^{3/4}}\, i d\theta$$

Estimating

$$|\mathcal{J}_2| \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\cos\theta - \frac{\eta\cos\left(\frac{\theta}{2} - \frac{\delta}{2}\right)}{(A^2\tau^2 + 2A\tau\cos\theta + 1)^{1/4}}} \frac{d\theta}{(A^2\tau^2 + 2A\tau\cos\theta + 1)^{3/4}}$$

$$\leq \pi e \tag{A33b}$$

Next we evaluate the first integral in (A32) in terms of known integrals. In light of the previous arguments (p. A4) the contour can be deformed into $C'\left(\int_{-\infty}^{(0+)}\right)$, which runs from $-\infty$ below the real axis is to $-\infty$ above the real axis circling the origin in a counterclockwise direction. There is a pole at the origin whose residue is 1. Then

$$\int_G e^{\alpha - \sqrt{\alpha}\eta} \frac{d\alpha}{\alpha} = 2\pi i + \int_{+\infty}^{0} e^{-\alpha + \eta i \sqrt{\alpha}} \frac{d\alpha}{\alpha}$$

$$+ \int_0^{\infty} e^{-\alpha - i\eta\sqrt{\alpha}} \frac{d\alpha}{\alpha}$$

$$= 2\pi i - 2i \int_0^{\infty} e^{-\alpha} \sin \eta\sqrt{\alpha} \frac{d\alpha}{\alpha}$$

or putting $\alpha = \beta^2$

$$\int_G = 2\pi i - 4i \int_0^{\infty} e^{-\beta^2} \sin \eta\beta \frac{d\beta}{\beta} \tag{A34}$$

The integral in (A34) is of the form of E_0 in (A21) and is

$$\pi E_0\left(\frac{\eta}{2}, \frac{1}{2}\right) = \frac{\pi}{2} \operatorname{erf}\left(\frac{\eta}{2}\right)$$

Thus:

$$\int_G e^{\alpha - \eta\sqrt{\alpha}} \frac{d\alpha}{\alpha} = 2\pi i \left\{ 1 - \operatorname{erf}\left(\frac{\eta}{2}\right) \right\} \tag{A35}$$

The solution (A9a) then is, from (A33)

$$u(\eta, \tau) = u_0 \left\{ 1 - \operatorname{erf} \frac{\eta}{2} \right\} + \mathcal{E} \qquad \text{where} \qquad \eta = \frac{x}{\sqrt{t}}, \quad \tau = t \tag{A36}$$

and from (A33a), (A33b)

$$|\mathcal{E}| < \frac{\tau\eta}{2} \left[4 + \pi e \right] \frac{1}{2\pi}$$

$$|\mathcal{E}| < 2\tau\eta = 2x\sqrt{t}$$

Similar results can also be obtained for the impulse signal.

APPENDIX B

Non-Linear Longitudinal Waves

A simple example of a non-linear longitudinal wave is a shock wave, which in a real fluid is of course not a discontinuity but a continuous steep variation. The effect of the non-linearity in maintaining the steepness and preventing viscous dispersion of the wave is vital, and is lost in any linearized theory.

The actual non-linear problems are very difficult and not, at present, capable of solution. However, the main effects can be shown by considering simplified problems. In the following sections an approximate treatment of weak non-stationary waves will be given.

Ba. Basic Equations for Transonic Flow

Consider one-dimensional longitudinal flow with the velocity $u(x,t)$. The basic assumption is that at $x=-\infty$ the flow is uniform and steady, slightly faster than the speed of sound;

$$u_{-\infty} = c^* + w_{-\infty} \tag{B1}$$

where the constant c^* is the speed of sound at M = 1. Under the conditions assumed, the flow is isentropic at $-\infty$ and the Bernoulli equation is

$$\frac{u^2}{2} + \frac{c^2}{\gamma-1} = \frac{c^{*2}}{2}\,\frac{\gamma+1}{\gamma-1} = \text{const.} \tag{B2}$$

The equations to be satisfied by the flow are continuity, momentum, energy, and the perfect gas law (cf. 1.11, 1.12, 1.13, 1.14)

$$\rho_t + (\rho u)_x = 0 \tag{B3a}$$

$$\rho u_t + \rho u u_x = -p_x + \frac{4}{3}(\mu u_x)_x \tag{B3b}$$

$$\rho c_v T_t + \rho u c_v T_x = -\rho \left[P\left(\frac{1}{\rho}\right)_t + u\, P\left(\frac{1}{\rho}\right)_x \right] + \frac{4}{3} \mu u_x^2 \qquad \text{(B3c)}$$

$$P = \rho \bar{R} T \qquad \text{(B3d)}$$

The assumptions of no heat conduction, no heat addition to the system, and a perfect gas have been made. The first step is the derivation of suitable equations from (B3).

Derivation of a Bernoulli equation, including viscosity.

For convenience introduce a velocity potential $\Phi(x,t)$ such that

$$u(x,t) = \Phi_x \qquad \text{(B4)}$$

Then integration of the momentum equation (B3b) from $-\infty$ to x yields

$$\Phi_t + \frac{1}{2}\Phi_x^2 = -\int_{-\infty}^{x} \frac{1}{\rho}\frac{\partial P}{\partial \xi} d\xi + \frac{4}{3}\int_{-\infty}^{x} \frac{1}{\rho}\frac{\partial}{\partial \xi}\left(\mu \frac{\partial^2 \Phi}{\partial \xi^2}\right) d\xi + \frac{1}{2} u_{-\infty}^2 \qquad \text{(B5)}$$

The right hand side of (B5) can be integrated under certain perturbation assumptions. For transonic flow assume that all the quantities differ slightly from their values at M = 1. Thus

$P = P^*(1+p^*)$ where $c^{*2} = \gamma \bar{R} T^* = \dfrac{\gamma P^*}{\rho^*}$ is defined from the condition at $x = -\infty$

$\rho = \rho^*(1+s^*)$

$T = T^*(1+\theta^*)$

$\mu = \mu^*(1+\alpha^*)$ where $\mu^* = \mu(T^*)$ (B6)

and where all the quantities $p^*, s^*, \theta^*, \alpha^* << 1$. Further introduce a perturbation potential $\varphi(x,t)$ such that

$$\Phi_x(x,t) = c^* + \varphi_x(x,t) \qquad \text{(B7)}$$

where $\varphi_x \ll c^*$

Then using (B1), (B6), and (B7) and neglecting higher order terms we approximate (B5) by

$$p^*(x,t) = p_{-\infty} + \frac{\gamma}{c^{*2}} \left\{ \frac{4}{3} \nu^* \varphi_{xx} - \varphi_t + c^* (w_{-\infty} - \varphi_x) \right\} \qquad \text{(B8)}$$

(B8) is the required relationship and expresses the pressure at any point in terms of the derivatives of the perturbation potential φ.

<u>Derivation of an equation of motion in terms of the perturbation potential</u> φ :

Multiplying the momentum equation (B3b) by u yields

$$u u_t + u^2 u_x = -\frac{u}{\rho} p_x + \frac{4}{3\rho} u \left\{ \mu u_x \right\}_x \qquad \text{(B9)}$$

The continuity, energy and state equations of (B3) can be used to eliminate p_x since

$$\frac{\mu}{\rho} p_x = \frac{4}{3}(\gamma - 1) \frac{\mu}{\rho} u_x^2 - \left(\frac{\gamma p}{\rho} \right) u_x - \frac{1}{\rho} p_t \qquad \text{(B10)}$$

Then (B9) and (B10) are combined as

$$u u_t + \left(u^2 - \frac{\gamma p}{\rho} \right) u_x = \frac{1}{\rho} p_t + \frac{4}{3} \frac{u}{\rho} (\mu u_x)_x - \frac{4}{3}(\gamma - 1) \frac{\mu}{\rho} u_x^2 \qquad \text{(B11)}$$

Introducing the perturbation assumptions (B6) and (B7) in (B11) and neglecting smaller terms, we have

$$c^* \varphi_{xt} + \left\{ 2c^* \varphi_x - a^* (p^* - s^*) \right\} \varphi_{xx} = \frac{c^{*2}}{\gamma} p_t^* \qquad \text{(B12)}$$

In order to obtain an equation for φ, we eliminate p_t^* from (B12) by using the Bernoulli equation (B8) and we relate p^* and s^* by using the energy equation (B3c). Introducing the perturbation assumptions in the energy equations

$$\left(\frac{\partial}{\partial t}+c^*\frac{\partial}{\partial x}\right)\left(\frac{p^*}{\gamma}-s^*\right)=\frac{4}{3}\frac{\nu^*}{c^{*2}}(\gamma-1)\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2 \qquad \text{(B13a)}$$

or integrating

$$\frac{p^*}{\gamma}-s^*=f(x-c^*t)+J(x,t) \qquad \text{(B13b)}$$

where J = integral of the dissipation, and $J(-\infty,t)=0$

The flow is isentropic at $x=-\infty$ so that $f=0$ and

$$p^*_{-\infty}-s^*_{-\infty}=-\frac{\gamma+1}{c^*}w_{-\infty} \qquad \text{(B14)}$$

Hence

$$p^*-s^*=\frac{\gamma-1}{c^{*2}}\left\{\frac{4}{3}\nu^*\varphi_{xx}-\varphi_t-a^*\varphi_x\right\} \qquad \text{(B15)}$$

Then using (B8) and (B15), we obtain from (B12) the following basic equation for $\varphi(x,t)$.

$$\varphi_{tt}+2c^*\varphi_{xt}+\left\{(\gamma+1)c^*\varphi_x+(\gamma-1)\varphi_t-c^{*2}J\right\}\varphi_{xx}=\frac{4}{3}\nu\left\{c^*\varphi_{xxx}+\varphi_{xxt}\right\} \qquad \text{(B16)}$$

Non-linear terms which allow for both the steepening of the wave $\left(u\frac{\partial u}{\partial x}\right)$ and variations in the local speed of sound are present in this equation. It should be remembered that the equation is expressed in a system of coordinates where the flow is slightly supersonic at $x=-\infty$. Notice also that the dissipation integral does not occur in the right hand side but only in the lower order terms.

Bb. Transonic Flow Through a Shock Wave

Steady-state solution: For a shock wave which is already developed $\varphi = \varphi(x)$ and (B16) becomes

$$(\gamma+1)\varphi_x \varphi_{xx} - c^* J(\varphi) = \frac{4}{3}\vartheta^* \varphi_{xxx} \tag{B17a}$$

where

$$J\{\varphi\} = \frac{4}{3}\frac{\vartheta^*}{c^{*2}}(\gamma-1)\int_{-\infty}^{x} \varphi_{\xi\xi}^2 \, d\xi \tag{B17b}$$

It is now assumed that the dissipation in (B17a) is small enough to be neglected and (B17a) is then integrated from $-\infty$ to x, to give

$$w^2 - w_{-\infty}^2 = \bar{\vartheta}\frac{dw}{dx} \quad \text{where} \quad \frac{8\vartheta^*}{3(\gamma+1)} = \bar{\vartheta} \tag{B18}$$

(B18) may also be integrated and if $x = 0$ when $w = 0$ we obtain

$$\frac{w}{w_{-\infty}} = -\tanh\left(\frac{w_{-\infty} x}{\bar{\vartheta}}\right) \tag{B19}$$

Also we have the potential

$$\varphi(x) = -\bar{\vartheta}\log\cosh\left(\frac{w_{-\infty} x}{\bar{\vartheta}}\right) \tag{B20}$$

if $\varphi(0) = 0$

(B19) represents a velocity distribution which is supersonic at $x = -\infty$ and subsonic at $x = \infty$, with a continuous variation which is most rapid in the neighborhood of the origin $x = 0$. For smaller values of ϑ^* this region of rapid change becomes increasingly narrower and approaches zero as $\vartheta^* \to 0$.

The important effect of the non-linearity in allowing a transition from $w > 0$, (supersonic flow), to $w < 0$, (subsonic flow) is thus shown. This transition is missing in any linearized theory.

As a check on the approximations the order of magnitude of the dissipation integral (B17b) can be found using the solution (B18). We obtain

$$J(x) = (\gamma-1)\,\frac{w_{-\infty}^{\;3}}{c^{*3}}\left\{\frac{2}{3} + \tanh\frac{w_{-\infty}x}{\vartheta} - \frac{1}{3}\tanh^3\frac{w_{-\infty}x}{\vartheta}\right\}$$

so that the total dissipation is

$$J(\infty) = \frac{4}{3}(\gamma-1)\left(\frac{w_{-\infty}}{c^*}\right)^3 \sim \left(M_{-\infty}^2 - 1\right)^3 \qquad \text{(B21)}$$

This indicates that the dissipation can be neglected if the Mach number ahead of the wave is sufficiently close to 1. Actually the steady state problem including the viscosity and dissipation can be solved for waves of arbitrary strength (See Ref 16). However the viscosity is assumed to be constant in that derivation. For a strong wave the variations of viscosity and heat conduction with temperature, and the dissipation all become important. The weak wave assumption is the only one consistent with assuming constant viscosity.

<u>An approximate non-steady solution:</u> In this approximation the integral of the dissipation is again neglected. Further, in order to simplify the problem, the assumption is made that

$$\varphi_t \ll c^* \varphi_x \qquad \text{(B22)}$$

so that certain terms in (B16) may be neglected. This is an approximation which becomes better for large t when the solution will be shown to approach the steady state. (B16) is thus approximated by

$$2\varphi_{xt} + (\gamma+1)\varphi_x\varphi_{xx} = \frac{4}{3}\vartheta^*\varphi_{xxx} \qquad \text{(B23)}$$

Integration of (B23) from $-\infty$ to x gives

$$\varphi_t + \frac{\gamma+1}{4}\left(\varphi_x^{\,2} - w_{-\infty}^{\,2}\right) = \frac{2\nu^*}{3}\,\varphi_{xx} \qquad \text{(B24)}$$

since $\varphi_x = w_{-\infty}$, $\varphi_{xx}, \varphi_t \to 0$ at $x = -\infty$. (B24) is a non-linear equation for which the general solution of the initial value problem for the domain $(-\infty < x < \infty,\ t \geqslant 0)$ can be given. Only a special example is treated here which is the diffusion by viscosity of an initially sharp wave front. The initial choice is the solution for a weak wave without viscosity and then at $t = 0$ the viscosity is introduced. The initial conditions are

$$\begin{aligned} \varphi(x,0) &= +\,w_{-\infty}\,x \qquad & x < 0 \\ &= -\,w_{-\infty}\,x \qquad & x > 0 \end{aligned} \qquad \text{(B25)}$$

The solution is found by assuming

$$\varphi(x,t) = F\left\{\theta(x,t)\right\} \qquad \text{(B26)}$$

where $\theta(x,t)$ is to satisfy a linear equation and the function F must be determined. Such an assumption about the solution is suggested by similarity solutions of (B24). Under the assumption (B26), (B24) becomes

$$F'(\theta)\,\theta_t + \frac{\gamma+1}{4}\left\{F'^{2}(\theta)\,\theta_x^{2} - w_{-\infty}^{2}\right\} = \frac{2\nu^*}{3}\left\{F''(\theta)\,\theta_x^{2} + F'(\theta)\,\theta_{xx}\right\} \qquad \text{(B27)}$$

(B27) is satisfied if $\frac{\gamma+1}{4}F'^{2} = \frac{2\nu^*}{3}F''$ (B28)

and

$$\theta_t - \frac{1}{F'(\theta)}\,\frac{\gamma+1}{4}\,w_{-\infty}^{2} = \frac{2\nu^*}{3}\,\theta_{xx} \qquad \text{(B29)}$$

The general solution of (B28) is

$$F(\theta) = c_2 - \bar{\nu}\,\log\left\{c_1 - \frac{1}{\bar{\nu}}\,\theta(x,t)\right\}$$

The constants c_2, c_1 may be chosen by having $\varphi \rightarrow \theta$ as $\frac{\gamma+1}{4} \rightarrow 0$, and having $F=0$ where $\theta = 0$. Then

$$F(\theta) = -\bar{\nu} \log \left(1 - \frac{1}{\bar{\nu}} \theta(x,t)\right) \tag{B30}$$

$$F'(\theta) = \frac{1}{1 - \frac{1}{\bar{\nu}} \theta(x,t)} \tag{B31}$$

and the linear equations to be satisfied by θ is a modified heat equation

$$\theta_t - \frac{\gamma+1}{4}\left(1 - \frac{1}{\bar{\nu}}\theta\right) = \frac{2\nu^*}{3}\theta_{xx} \tag{B32}$$

The boundary conditions for (B32) are found from the formulas relating θ, φ through the known function F:

$$\varphi(x,t) = -\bar{\nu} \log\left\{1 - \frac{1}{\bar{\nu}}\theta(x,t)\right\} \tag{B33a}$$

$$\theta(x,t) = \bar{\nu}\left\{1 - \exp\left[-\frac{1}{\bar{\nu}}\varphi(x,t)\right]\right\} \tag{B33b}$$

Thus using (B25) the initial conditions for (B32) are

$$\theta(x,0) = \bar{\nu}\left\{1 - e^{-\frac{W_{-\infty}x}{\bar{\nu}}}\right\} \qquad x<0 \tag{B34a}$$

$$= \bar{\nu}\left\{1 - e^{+\frac{W_{-\infty}x}{\bar{\nu}}}\right\} \qquad x>0 \tag{B34b}$$

The solution to (B32) satisfying the boundary conditions (B34) may be obtained by Laplace transformation or other methods as

$$\theta(x,t) = \bar{\nu}\left\{1 + \frac{1}{\bar{\nu}} e^{-\frac{(\gamma+1)W_{-\infty}^2 t}{2\bar{\nu}}} \int_0^\infty e^{-\tau t} \frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)W_{-\infty}^2}{2\bar{\nu}} + \tau\right)} d\tau \right.$$

$$\left. - 2\cosh\frac{W_{-\infty}x}{\bar{\nu}}\right\} \qquad x>0 \tag{B35}$$

It may be verified that (B35) satisfies the boundary condition (B34b) by virtue of the formula

$$\frac{w_{-\infty}x}{\bar{\nu}}\frac{1}{\pi}\int_0^\infty \frac{\cos\sqrt{\frac{3\tau}{2\nu^*}}\,x}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_{-\infty}^2}{2\bar{\nu}}+\tau\right)}\,d\tau = e^{-\frac{w_{-\infty}x}{\bar{\nu}}} \qquad x>0 \tag{B36}$$

Thus the potential velocity distribution and $x>0$ are obtained for and $\varphi_x(-x,t)=-\varphi_x(x,t)$, $\varphi_t(-x,t)=\varphi_t(x,t)$.

$$\varphi(x,t)=-\bar{\nu}\log\left\{2\cosh\frac{w_{-\infty}x}{\bar{\nu}}-\frac{1}{\bar{\nu}\pi}e^{-\frac{(\gamma+1)w_{-\infty}^2 t}{2\bar{\nu}}}\int_0^\infty e^{-\tau t}\frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_{-\infty}^2}{2\bar{\nu}}+\tau\right)}d\tau\right\} \tag{B37a}$$

$$\varphi_x=-w_{-\infty}\frac{\sinh\frac{w_{-\infty}x}{\bar{\nu}}+\frac{1}{2\pi}e^{-\frac{(\gamma+1)w_{-\infty}^2 t}{2\bar{\nu}}}\int_0^\infty e^{-\tau t}\frac{\sin\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_{-\infty}^2}{2\bar{\nu}}+\tau\right)}d\tau}{\cosh\frac{w_{-\infty}x}{\bar{\nu}}-\frac{1}{2\bar{\nu}\pi}e^{-\frac{(\gamma+1)w_{-\infty}^2 t}{2\bar{\nu}}}\int_0^\infty e^{-\tau t}\frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_{-\infty}^2}{2\bar{\nu}}+\tau\right)}d\tau} \tag{B37b}$$

$$\varphi_t=-w_{-\infty}^2\sqrt{\frac{2\nu^*}{3\pi t}}\frac{e^{-\frac{(\gamma+1)w_{-\infty}^2 t}{2\bar{\nu}}-\frac{x^2}{t\bar{\nu}(\gamma+1)}}}{2\cosh\frac{w_{-\infty}x}{\bar{\nu}}-\frac{1}{\pi\bar{\nu}}e^{-\frac{(\gamma+1)w_{-\infty}^2 t}{2\bar{\nu}}}\int_0^\infty e^{-\tau t}\frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_{-\infty}^2}{2\bar{\nu}}+\tau\right)}d\tau} \tag{B37c}$$

This solution (B37) approaches the steady-state solution (B30) very rapidly because of the exponential damping with time. The ratio $\frac{\varphi_t}{c^*\varphi_x}$ may be computed from the solution (B37) and it can be seen that for $t>0$ this term is very small except when $x=0$, where $\varphi_x=0$. For any fixed $x\neq 0$ the ratio becomes very small after some time.

Thus one simple example has been given where the steady state of the shock wave is the limit of special non-steady solution. The problem had to be simplified greatly in order to make an analytical treatment possible. However, the essential features of non-linearity and viscosity were retained.

APPENDIX C

Non-Linear Effects in Transversal Waves (One-Dimensional)

In this appendix non-linear effects which arise in a transversal motion are studied briefly. In the cases which are studied here the important non-linear effects are the dissipative ones, due to viscosity and heat conduction. As an example we can consider the flow generated by the transversal (i.e. parallel to x) motion of an infinite plane $(y=0)$. Then the component of velocity u parallel to the wall will be produced for the most part by the transverse shearing effect (as in § 2.3). However the dissipative effects, involving squared terms such as u_y^2, v_y^2, produce viscous heating important near the wall. This heating sets up a pressure gradient p_y and a longitudinal velocity wave v is produced. Thus, the dissipation couples longitudinal and transversal waves.

For a more complete discussion, consider the following equations of motion, which can be derived from the fundamental equations (1.11-1.14) by considering all functions to depend on (y,t)

$$\rho u_t + \rho v u_y = (\mu u_y)_y \qquad x\text{-momentum} \quad \text{(a)}$$

$$\rho v_t + \rho v v_y = \frac{4}{3}(\mu v_y)_y - p_y \qquad y\text{-momentum} \quad \text{(b)}$$

$$\rho_t + (\rho v)_y = 0 \qquad \text{Continuity} \quad \text{(c)} \quad \text{(C1)}$$

$$\rho c_p T_t + \rho c_p v T_y - p_t - v p_y = (k T_y)_y + \mu u_y^2 + \frac{4}{3}\mu v_y^2 \qquad \text{Energy} \quad \text{(d)}$$

$$p = \bar{R} \rho T \qquad \text{State} \quad \text{(e)}$$

where in general $c_p = c_p(T)$, $k = k(T)$, $\mu = \mu(T)$. A typical problem for the system (C1) is a determination of the flow field for

$y > 0$ when the plane $y = 0$ is given a prescribed motion parallel to itself and the temperature at the wall T_w (or heat flow through the wall) is also prescribed with time. In this case transversal velocity waves, longitudinal velocity and pressure waves, and temperature waves all arise.

An exact solution of the problem is not feasible so that various approximate procedures must be used. One method, somewhat analogous to the classical boundary-layer theory, has been carried out by S. Corrsin (Ref. 55) for the plate suddenly starting in motion with a constant velocity. In this work the main attention is concentrated on the velocity profile parallel to the plate. P is assumed constant and $v << u$ in the first approximation. The vertical velocity v is found from the continuity equation where the variations in ρ are known. The results indicate that the velocity profiles become more nearly linear functions of $\left(\frac{y}{\sqrt{\nu t}}\right)$ as the speed of starting becomes 4 or 5 times the speed of sound.

We take a different viewpoint here in accordance with our previous work. We attempt again to get the overall picture for a much simplified problem. First let us eliminate the temperature by means of the equation of state. Then equations (C1d) and (C1e) can be combined to read

$$P_t + v P_y - \left(\frac{\gamma P}{\rho}\right) \left(\rho_t + \rho_y\right) = (\gamma-1)\,\mu \left\{ u_y^2 + \frac{4}{3} v_y^{\,2} \right\} + (\gamma-1)\left[k\left(\frac{P}{\bar{R}\rho}\right)_y \right]_y \qquad \text{(C2)}$$

It can be noted that if the dissipation u_y^2 is omitted from (C2), the three equations (C1b), (C1c), (C2) form a system for (P, ρ, v). Then u could be found afterwards from (C1a). Thus this dissipation term, in a sense, is the vital coupling between the longitudinal and transversal wave systems. Then, for simplification we adopt a certain "linearizing" process which retains this term but eliminates all other non-linear difficulties. Assume

i) $\rho = \rho_0 \; (1+s)$ $\qquad s << 1$

ii) $P = P_0 \; (1+p)$ $\qquad p << 1$

iii) $\mu \approx \mu_0$ $\qquad k \equiv 0$ $\qquad$ (no heat conduction)

iv) u, v $\qquad$ both small

$$v u_y << u_t \qquad v v_y << v_t$$

Then the system (C1) together with (C2) becomes the following system of "linearized" equations

$$u_t = \nu u_{yy} \tag{a}$$

$$v_t = \frac{4}{3} \nu v_{yy} - \frac{c^2}{\gamma} p_y \tag{b}$$

(C3)

$$v_y = - s_t \tag{c}$$

$$p_t - \gamma s_t = \frac{\gamma(\gamma-1)\nu}{c^2} \; {u_y}^2 \tag{d}$$

With this simplification u is determined as $u(y,t)$, first, as in Ref. 55, under zero initial conditions and the boundary condition $u(0,t) = f(t) \quad t > 0$. When u is known the right hand side of (C3d) is a known function, representing the distribution of dissipation due to the transverse motion u. Thus, under the proper boundary conditions v, p, s can be found from (C3b), (C3c), (C3d).

Equation (C3d) can be integrated with respect to time to give

$$p(y,t) = \gamma s(y,t) + \frac{\gamma(\gamma-1)\nu}{c^2} \int_0^t u_y^2 (y,\tau)\, d\tau \tag{C4}$$

Equation (C4) shows the deviation from an isentropic process due to the dissipation in this approximation. Elimination of the pressure from (C3b) leads to the following system for v, s

$$\frac{4}{3}\nu v_{yy} - v_t - c^2 s_y = (\gamma-1)\nu \frac{\partial}{\partial y} \int_0^t u_y^2 (y,\tau)\, d\tau \qquad \text{(a)}$$

(C5)

$$v_y + s_t = 0 \qquad \text{(b)}$$

When v, s are found from (C3), $p(y,t)$ is given by equation (C4). System (C5) is identical with the system whose fundamental solution is discussed in Appendix D. This permits an interpretation of the right hand side of (C5a).

$$-Y(y,t) = (\gamma-1)\nu \frac{\partial}{\partial y} \int_0^t u_y^2 (y,\tau)\, d\tau \tag{C6}$$

as a variable external force in the y-direction. The solution to (C5) can be obtained by application of the integral formula (D33) of Appendix D. An equation for v itself can be obtained also by eliminating s from (C5a)

$$\frac{4}{3}\nu v_{yyt} - v_{tt} + c^2 v_{yy} = (\gamma-1)\nu \frac{\partial}{\partial y} u_y^2 (y,t) \tag{C7}$$

APPENDIX D

Construction of Fundamental Solutions.

In §2.1 an intuitive discussion of the concept of fundamental solution was given. In this appendix the fundamental solutions of the linearized viscous equations are constructed for various cases. They are found as certain integral representations, and approximate formulas are obtained in special cases. A rather general theorem on fundamental solutions is proved in the course of the analysis. In addition, it is indicated how solutions for boundary-value problems may be obtained by using the fundamental solutions in integral formulas. The basic ideas in this appendix are due to A. V. Pleijel. See also Ref. 21, 23, and 60.

Da. One-Dimensional Non-Stationary Waves

The equations of motion in one dimension can be written in terms of dimensionless variables and functions as:

$$u_{xx} - u_t - s_x = -X(x,t) \tag{D1a}$$

$$u_x + s_t = 0 \tag{D1b}$$

The function X is given for all x and $t \geq 0$ and is assumed to be zero for $t < 0$. With the aid of a fundamental solution, $u(x,t)$ and $s(x,t)$, the solutions to D1 in $(-\infty \leq x \leq \infty,\ t \geq 0)$ with homogeneous boundary conditions, may be expressed as integrals involving the given function X. More explicitly, the conditions are:

Zero initial conditions:

$$u(x,0) = 0 \tag{D2a}$$

$$s(x,0) = 0 \tag{D2b}$$

Conditions of damping at ∞

$$u(\pm\infty, t) = 0 \tag{D3a}$$

$$s(\pm\infty, t) = 0 \tag{D3b}$$

$$u(x,\infty)=0 \tag{D4a}$$

$$S(x,\infty)=0 \tag{D4b}$$

The function X is assumed to have properties which make these conditions possible.

Then a fundamental solution of this system in the half-plane $t>0$ is defined as a pair of functions $\Gamma(x,t;\xi,\tau)$, $S(x,t;\xi,\tau)$ such that

$$u(x,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \Gamma(x,t;\xi,\tau)\; X(\xi,\tau)\; d\tau\, d\xi \tag{D5a}$$

$$s(x,t) = \int_{-\infty}^{\infty}\int_{0}^{\infty} S(x,t;\xi,\tau)\; X(\xi,\tau)\; d\tau\, d\xi \tag{D5b}$$

is the solution to (D1) under the conditions (D2), D3), and (D4). It follows from the homogeneity of the system (D1) that Γ and S can depend only on the differences $t-\tau$, $x-\xi$. Further, it is assumed that Γ and S are identically zero for $\tau>t$. This means that the solution at a time t does not depend on the behavior of the function $X(\xi,\tau)$ at a time later than t. Thus, we may rewrite (D5) as

$$u(x,t) = \int_{-\infty}^{\infty}\int_{0}^{t} \Gamma(x-\xi,t-\tau)\, X(\xi,\tau)\; d\tau\, d\xi \tag{D6a}$$

$$s(x,t) = \int_{-\infty}^{\infty}\int_{0}^{t} S(x-\xi,t-\tau)\, X(\xi,\tau)\, d\tau\, d\xi \tag{D6b}$$

The problem of finding Γ and S for the partial differential equations (D1) can be reduced to the problem of finding the fundamental solution of an ordinary differential equation if the time variable is eliminated from (D1) by the use of Laplace transforms. In the following we denote Laplace transforms by bars over the symbols. Thus:

$$\mathcal{L}\{f\} = \bar{f}(x;\sigma) = \int_{0}^{\infty} e^{-\sigma t} f(x,t)\,dt \tag{D7}$$

An application of this transformation and of the convolution theorem (Ref. 2) to (D6) yields:

$$\bar{u}(x;\sigma) = \int_{-\infty}^{\infty} \bar{\Gamma}(x-\xi;\sigma)\, \bar{X}(\xi;\sigma)\, d\xi \tag{D8a}$$

$$\bar{s}(x;\sigma) = \int_{-\infty}^{\infty} \bar{S}(x-\xi;\sigma)\, \bar{X}(\xi;\sigma)\, d\xi \tag{D8b}$$

An application of the transformation to equations (D1), taking account of the zero initial condition (D2), gives the following subsidiary equations:

$$\frac{d^2\bar{u}}{dx^2} - \sigma\bar{u} - \frac{d\bar{s}}{dx} = -\bar{X} \tag{D9a}$$

$$\frac{d\bar{u}}{dx} + \sigma\bar{s} = 0 \tag{D9b}$$

and boundary conditions (D3) become

$$\bar{u}(\pm\infty;\sigma) = 0 \tag{D10a}$$

$$\bar{s}(\pm\infty;\sigma) = 0 \tag{D10b}$$

Now we can obtain the following ordinary differential equation for $u(x;\sigma)$ from (D9):

$$\frac{\sigma+1}{\sigma}\,\frac{d^2\bar{u}}{dx^2} - \sigma\bar{u} = -\bar{X}(x;\sigma) \tag{D11}$$

The form of (D8a) indicates that $\bar{\Gamma}(x-\xi)$ is just the fundamental solution of (D11) according to the usual definition (Ref. 1, Vol. II, p. 302). We have

$$\bar{\Gamma}(x-\xi;\sigma) = \frac{1}{2\sqrt{1+\sigma}}\, e^{-\frac{\sigma|x-\xi|}{\sqrt{1+\sigma}}} \tag{D12}$$

Thus the determination of the fundamental solution Γ depends on an inversion of $\bar{\Gamma}$. This is easily expressed by the complex inversion formula (Ref. 2) as

$$\Gamma(x-\xi, t-\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\sigma(t-\tau)}}{2\sqrt{1+\sigma}} e^{-\frac{\sigma|x-\xi|}{\sqrt{1+\sigma}}} d\sigma \qquad \text{(D13)}$$

so that $\Gamma \equiv 0 \qquad \tau > t$

The relationship (D9b) is valid for the fundamental solution so that

$$S(x-\xi, t-\tau) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\sigma(t-\tau)}}{2(1+\sigma)} e^{-\frac{\sigma|x-\xi|}{\sqrt{1+\sigma}}} \operatorname{sign}(x-\xi)\, d\sigma \qquad \text{(D14)}$$

Db. Integral Formulas. Boundary-Value Problems.

Formulas (D6) show how a solution of the non-homogeneous system (D1) is constructed if the initial conditions are zero and if the domain is unbounded ($-\infty < x < \infty$). In order to solve more general problems with non-zero initial conditions and with other boundary values, more general integral formulas will have to be derived. This is carried out below by an extension to the system (D1) of the method used for second order parabolic equations (Ref. 56, p. 129; Ref. 3, Vol. III, p. 130). This method involves in particular an analysis of the singularity of the fundamental solution. An integral formula is proved for a rectangle. This leads immediately to a representation formula for the solution in a half-plane ($t > t_1$, $-\infty < x < \infty$). In particular, this formula includes (D6) as a special case. A representation formula for a quadrant is also found by use of a reflection principle. By a similar procedure, integral formulas could be proved for the higher-dimensional cases whose fundamental solutions are constructed in this appendix. This has not been carried out in the present report; see, however, Ref. 21 for the incompressible case, and Ref. 23 and 60 for the compressible case.

Consider the original system

$$L(u,s) = u_{xx} - u_t - s_x = -X(x,t) \tag{D1a}$$

$$u_x + s_t = 0 \tag{D1b}$$

and the adjoint system.

$$\bar{\bar{L}}(w,\sigma) = w_{xx} + w_t + \sigma_x = 0 \tag{D15a}$$

$$w_x + \sigma_t = 0 \tag{D15b}$$

Then

$$wL(u,s) - u\bar{\bar{L}}(w,\sigma) = w(u_{xx} - u_t - s_x) - u(w_{xx} + w_t + \sigma_x) = -wX$$

$$= \Big\{ w\,[u_x - s] - u\,[w_x + \sigma] \Big\}_x (-wu)_t - (s\sigma)_t = -wX \tag{D16}$$

by using (D1b) and (D15b). The equation (D16) is now integrated over the domain G' consisting of the rectangle bounded by x_1, x_0, t_1, t' (Fig. D1). Thus, using (ξ,τ) as variables of integration,

$$\iint_{G'} wL(u,s) - u\bar{\bar{L}}(w,\sigma)\,d\tau\,d\xi = -\iint_{G'} wX\,d\tau\,d\xi \tag{D17}$$

or

$$\int_{t_1}^{t'} \left\{ w\left[u_\xi - s\right] - u\left[w_\xi + \sigma\right]\right\}\Bigg|_{x_1}^{x_0} d\tau - \int_{x_1}^{x_0} (wu + s\sigma)\Bigg|_{t_1}^{t'} d\xi = -\iint_{G'} wX\,d\tau\,d\xi \tag{D17'}$$

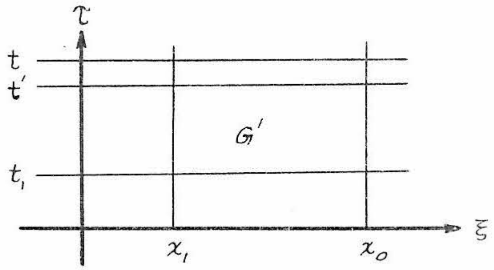

Figure D.1

A representation formula for the solution u at point (x,t) in terms of certain boundary values can now be obtained from (D17'). To do this w is taken to be the fundamental solution of (D1) associated with the point (x,t), $\Gamma(x-\xi, t-\tau)$ as given by (D13), and σ to be $S(x-\xi, t-\tau)$ of (D14). The formula (D17) still holds since Γ, S considered as functions of (ξ,τ) are solutions to the adjoint system (D15). As will be shown, $\Gamma(x-\xi, t-\tau)$ has a singularity at $x=\xi$ on the line $\tau=t$ and for that reason the integrals in (D17) are taken on the line $t'<t$. The limiting procedure $t'\to t$ is needed to represent the solutions $u(x,t)$ and this procedure depends very much upon the

singularity of Γ. By methods similar to those applied to (A9a') in Appendix A an evaluation of (D13) for small $t-\tau$ can be obtained. We have

$$\Gamma(x-\xi, t-\tau) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t-\tau}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} + \mathcal{E}_1 \tag{D18}$$

where $|\mathcal{E}_1| < C\sqrt{t-\tau}$; C is a positive constant

Now consider u_p defined by

$$u_p = \int_{x_1}^{x_0} \Gamma(x-\xi, t-t')\, u(\xi, t')\, d\xi \tag{D19a}$$

$$= \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{t-t'}} \int_{x_1}^{x_0} e^{-\frac{(x-\xi)^2}{4(t-t')}} u(\xi, t')\, d\xi + \int_{x_1}^{x_0} \mathcal{E}_1\, u(\xi, t')\, d\xi \tag{D19b}$$

As $t' \to t$ the second integral in (D19b) vanishes while the first can be evaluated. We have

$$\left| \int_{x_1}^{x_0} \mathcal{E}_1\, u(\xi, t')\, d\xi \right| < C\sqrt{t-t'} \int_{x_1}^{x_0} |u(\xi, t')|\, d\xi \to 0 \qquad \text{as} \qquad t' \to t$$

assuming $\int_{x_1}^{x_0} |u|\, d\xi$ exists. Also, introducing the transformation

$$\xi = x + 2\lambda\sqrt{t-t'} \tag{D20}$$

into the first integral of (D19b), we obtain

$$\lim_{t' \to t} u_p = \frac{1}{\sqrt{\pi}} \lim_{t' \to t} \int_{\lambda_1}^{\lambda_2} u\,(x + 2\lambda\sqrt{t-t'}, t')\, e^{-\lambda^2}\, d\lambda \tag{D21}$$

The limits λ_1, λ_2 have different values depending on the location of x. For

$x_1 < x < x_0$, as $t' \to t$; $\lambda_1 \to -\infty$, $\lambda_2 \to +\infty$ so that

$$u_p \to \frac{1}{\sqrt{\pi}} u(x,t) \int_{-\infty}^{\infty} e^{-\lambda^2}\, d\lambda = u(x,t) \tag{D22}$$

For $x < x_1 < x_0$ or $x_1 < x_0 < x$; $\lambda_1 \to \lambda_2 \to \pm\infty$ so that

$$u_p \to 0 \tag{D23}$$

Also if $x_1 = x$; $\lambda_1 \to 0$, $\lambda_2 \to +\infty$, and

$$u_p \to \frac{1}{2}\, u(x_1, t) \tag{D24}$$

The behavior of the fundamental solution S as $t' \to t$ must also be determined. A formula similar to (D18) can be obtained which shows that for small $t-\tau$

$$S(x-\xi, t-\tau) = \frac{1}{2}\, \mathrm{erfc}\left\{\frac{|x-\xi|}{2\sqrt{t-\tau}}\right\} + \text{error term} \tag{D25}$$

Taking the error term to be negligible as before (D19), we obtain

$$\lim_{t' \to t} S_p = \lim_{t' \to t} \frac{1}{2} \int_{x_1}^{x_0} \mathrm{erfc}\left\{\frac{|x-\xi|}{2\sqrt{t-t'}}\right\} s(\xi, t')\, d\xi \; \mathrm{sign}\,(x-\xi)$$

$$= \lim_{t' \to t} \left\{ + \frac{1}{2} \int_{x_1}^{x-\epsilon} \mathrm{erfc}\left\{\frac{|x-\xi|}{2\sqrt{t-t'}}\right\} s(\xi, t')\, d\xi + \frac{1}{2} \int_{x-\epsilon}^{x+\epsilon} \mathrm{erfc}\left\{\frac{|x-\xi|}{2\sqrt{t-t'}}\right\} s(\xi, t')\, d\xi \; \mathrm{sign}\,(x-\xi) \right.$$

$$\left. - \frac{1}{2} \int_{x+\epsilon}^{x_0} \mathrm{erfc}\left\{\frac{|x-\xi|}{2\sqrt{t-t'}}\right\} s(\xi, t')\, d\xi \right\}$$

$\left| \lim_{t' \to t}, S_p \right| \leqslant \epsilon s(x + A\epsilon, t)$ where ϵ is as small as we please,

since $\quad \mathrm{erfc}\, \infty = 0, \; |\mathrm{erfc}\, x| < 1$

Thus

$$S_p \to 0 \text{ as } t' \to t \tag{D26}$$

Using the results of (D22), (D23), and (D26) in the integral formula, one finds

$$\int_{t'}^{t} \left\{ \Gamma(x-\xi, t-\tau)\, [u_\xi - s] - u\, [\Gamma_\xi + S] \right\} \Bigg|_{\xi = x_1}^{\xi = x_0} d\tau - u_p + \int_{x_1}^{x_0} [\Gamma u + S s]_{\tau = t_1}\, d\xi = \iint_G \Gamma X\, d\tau\, d\xi \tag{D27}$$

A representation formula for the solution in the half-plane $t > 0$ is obtained from (D27) by letting $x_0 \to +\infty$, $x_1 \to -\infty$. Then the first integrals in (D27) vanish due to the exponential damping of the fundamental solutions as $\{x_1 \to -\infty\}$, and u_p is given by (D22). If at the same time $t_1 = 0$, (D27) becomes

$$u(x,t) = \int_{-\infty}^{\infty}\left\{\Gamma(x-\xi,t)\,u(\xi,0) + S(x-\xi,t)\,s(\xi,0)\right\}d\xi + \int_{-\infty}^{\infty}\int_{0}^{t}\Gamma(x-\xi,t-\tau)X(\xi,\tau)\,d\tau\,d\xi \quad \text{(D28)}$$

(D28) expresses $u(x,t)$ in the domain $(-\infty < x < \infty,\ t \geq 0)$ in terms of the initial values $u(x,0)$, $s(x,0)$ and the given function X. It should be noted that a knowledge of $s(x,0)$ is equivalent to a knowledge of $u_t(x,0)$ by (D1a).

u and u_t would be the initial data prescribed if a problem for u alone were to be solved. The equation in that case is

$$u_{xxt} + u_{xx} - u_{tt} = -X_t \quad \text{(D29)}$$

It should also be noted that the initial data required (u, u_t) are just those needed for the wave equation $u_{xx} - u_{tt} = F(x,t)$. This is consistent with a treatment of (D29) by means of the Laplace transformation. However, the solution of (D29) is very different from the solution of a wave equation in its dependence on the initial data. Equation (D28) shows that the solution $u(x,t)$ depends on the entire distribution of initial data $(-\infty < x < \infty)$ whereas the solution of a hyperbolic wave equation at a point P depends only on those initial data on a finite part of the line $t = 0$.

A representation formula for the domain $(0 \leq x < \infty,\ t \geq 0)$ can also be obtained from (D27) either by constructing a Green's function for the domain or by a process of reflection analogous to that used in potential theory. We will use the latter procedure. Let $x_1 = 0$, $x_0 \to \infty$, $t_1 = 0$ in (D27) and obtain

$$\int_0^t \left\{ \Gamma\,(x,t-\tau)\left[u_\xi\,(o,\tau)-s\,(o,\tau)\right]-u(o,\tau)\left[\Gamma_\xi\,(x,t-\tau)+S(x,t-\tau)\right]\right\} d\tau - u_p$$

$$+\int_0^\infty \left[\Gamma u + S s\right]_{\tau=0} d\xi = -\int_0^\infty d\xi \int_0^t \Gamma\,(x-\xi,t-\tau)\; X\,(\xi,\tau)\,d\tau \qquad \textbf{(D30)}$$

For $x > 0$, u_p is given by (D22), and (D30) becomes the integral formula

$$u\,(x,t) = \int_0^t \Gamma(x,t-\tau)\left[u_\xi\,(o,\tau)-s(o,\tau)\right]-u(o,\tau)\left[\Gamma_\xi\,(x,t-\tau)+S\,(x,t-\tau)\right]$$

$$+\int_0^\infty \left[\Gamma\,(x-\xi,t)\,u(\xi,o)+S(x-\xi,t)\;s\,(\xi,o)\right] d\xi$$

$$+\int_0^\infty d\xi \int_0^t \Gamma\,(x-\xi,t-\tau)\,X(\xi,\tau)\,d\tau \qquad \textbf{(D31a)}$$

(D31) is not a representation formula because too many conditions $(u\,,u_x\,,s)$ are required on the line $x=0$. Two of these may be eliminated by considering (D30) at the reflected point $(-x,t)$. Then $u_p \to 0$ from (D23), and (D30) becomes

$$0=\int_0^t \left\{\Gamma\,(-x,t-\tau)\;\left[u_\xi\,(o,\tau)-s\,(o,\tau)\right]-u(o,\tau)\left[\Gamma_\xi\,(-x,t-\tau)+S\,(-x,t-\tau)\right]\right\} d\tau$$

$$+\int_0^\infty \left[\Gamma\,(-x-\xi,t)\;u\,(\xi,o)+S\,(-x-\xi,t)\;\;s(\xi,o)\right] d\xi$$

$$+\int_0^\infty d\xi \int_0^t \Gamma(-x-\xi,t-\tau)\;X(\xi,\tau)\,d\tau \qquad \textbf{(D31b)}$$

But an inspection of (D13) and (D14) shows that Γ and S have the following properties

$$\Gamma(x,t-\tau) = \Gamma(-x,t-\tau) \tag{D32a}$$

$$S(-x,t-\tau) = -S(x,t-\tau) \tag{D32b}$$

$$\Gamma_{\xi}(-x,t-\tau) = -\Gamma_{\xi}(x,t-\tau) \tag{D32c}$$

Hence subtracting (D31b) from (D31a) we obtain,

$$\begin{aligned} u(x,t) = &-2\int_0^t \left\{ \Gamma_{\xi}(x,t-\tau) + S(x,t-\tau) \right\} u(o,\tau)\,d\tau \\ &+ \int_0^\infty \left\{ \Gamma(x-\xi,t) - \Gamma(-x-\xi,t) \right\} u(\xi,o)\,d\xi + \int_0^\infty \left\{ S(x-\xi,t) + S(-x-\xi,t) \right\} s(\xi,o)\,d\xi \\ &+ \int_0^\infty d\xi \int_0^t \left\{ \Gamma(x-\xi,t-\tau) - \Gamma(-x-\xi,t-\tau) \right\} X(\xi,\tau)\,d\tau \end{aligned} \tag{D33}$$

for $x > o, t \geqslant o$

(D33) is the required representation formula useful for a "piston" problem or mixed boundary value problem where the value of $u(o,t)$ is given for all $t > 0$.

The representation formulas thus give the solution and show what boundary values are needed for a given domain. It should be remarked that not much additional work is required if a domain more general than G_1 is considered, namely one having rather general curves for the side boundaries, instead of x_1 = constant, x_0 = constant.

Dc. Longitudinal Non-stationary Waves in Higher Dimensions

In this section the method of Da will be applied to cases of more space dimensions. In particular, we shall find the velocity potential φ of longitudinal waves in two and three space dimensions. The equations for the potential (cf. 1.52) written in terms of dimensionless variables and functions are

$$\Delta\varphi - \varphi_t - s = -\Xi\,(x_i, t) \tag{D34a}$$

$$\Delta\varphi + s_t = 0 \tag{D34b}$$

Δ = Laplace operator in two or three space dimensions.

The fundamental solution, as before, is assumed to depend only upon distance r, and time shift $t-\tau$. Here $r^2 = \sum_{i=1}^{n} (x_i - \xi_i)^2$ where $n=2$ or 3 depending on the number of space dimensions. Thus the fundamental solution is a pair of functions:

$$\Phi = \Phi(r, t-\tau)$$
$$S = S(r, t-\tau)$$

such that

$$\varphi(x_i, t) = \int_{-\infty}^{\infty} dx_i \int_0^t \Phi(r, t-\tau)\,\Xi(\xi_i, \tau)\,d\tau \tag{D35a}$$

$$s(x_i, t) = \int_{-\infty}^{\infty} dx_i \int_0^t S(r, t-\tau)\,\Xi(\xi_i, \tau)\,d\tau \tag{D35b}$$

is the solution to (D34) with homogeneous boundary conditions at ∞ and zero initial conditions. In (D35) $\int_{-\infty}^{\infty} dx_i$ represents integration over the entire two or three dimensional space. The introduction of Laplace transforms as in (Da) shows that $\overline{\Phi}(r;\sigma)$ is the fundamental solution of the partial differential equation:

$$\frac{1+\sigma}{\sigma}\,\Delta\bar{\varphi}-\sigma\bar{\varphi}=-\bar{\Xi}(x_i,\sigma) \tag{D36}$$

The requirements for a fundamental solution are similar to those for an ordinary differential equation. Hence, in <u>two space dimensions</u> the Laplace transforms of the fundamental solution are:

$$\bar{\Phi}(r;\sigma)=\frac{1}{2\pi}\,\frac{\sigma}{\sigma+1}\;K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right) \tag{D37a}$$

$$\bar{S}(r;\sigma)=-\frac{1}{2\pi}\left(\frac{\sigma}{1+\sigma}\right)^2 K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right) \tag{D37b}$$

where $\quad r^2=(x_1-\xi_1)^2+(x_2-\xi_2)^2$

The complex inversion formula yields

$$\Phi(r,t-\tau)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)}\,\frac{1}{2\pi}\,\frac{\sigma}{1+\sigma}\,K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right)d\sigma \tag{D38a}$$

and

$$S(r,t-\tau)=\frac{-1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)}\,\frac{1}{2\pi}\left(\frac{\sigma}{1+\sigma}\right)^2 K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right)d\sigma \tag{D38b}$$

In <u>three space dimensions</u> the Laplace transforms of the fundamental solution are

$$\bar{\Phi}(r;\sigma)=\frac{1}{4\pi r}\,\frac{1}{1+\sigma}\,e^{-\frac{\sigma r}{\sqrt{1+\sigma}}} \tag{D39a}$$

$$\bar{S}(r;\sigma)=-\frac{1}{4\pi r}\,\frac{\sigma}{(1+\sigma)^2}\,e^{-\frac{\sigma r}{\sqrt{1+\sigma}}} \tag{D39b}$$

where

$$r^2=(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2$$

The inversion formula gives

$$\Phi(r,t-\tau)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)}\,\frac{1}{4\pi r}\,\frac{1}{1+\sigma}\,e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}\,d\sigma \tag{D40a}$$

$$S(r,t-\tau)=\frac{-1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)}\,\frac{1}{4\pi r}\,\frac{\sigma}{1+\sigma}\,e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}\,d\sigma \tag{D40b}$$

Dd. A General Theorem about Fundamental Solutions

We would now like to find the fundamental solution of the full system of linearized equations for various cases (e.g. (1.37)). These equations are considerably more complicated than those treated in (Da) and (Db). In order to simplify the work, a general theorem will be proved in this section which reduces the problem of finding the fundamental solution for a typical system to that of finding the fundamental solution of a single simpler equation. This theorem will be applied to various cases in the following sections.

The theorem is derived for solutions defined in an n-dimensional vector space R_n whose points are denoted by x, ξ. Let M_1 and M_2 be two linear differential matrix operators defined on vector function over R_n. Then the problem is the determination of the fundamental matrix (tensor) $\Gamma(x,\xi)$ for the differential equation

$$(a M_1 - b M_2 - k^2)\, u = - X(x) \tag{D41}$$

where: a, b, k are constants

$X(x)$ is a given vector function defined over R_n and vanishing suitably at infinity.

The solution $u(x)$ to (D41) is also a vector function and the fundamental solution is defined by the requirement that the solution satisfying homogeneous boundary conditions is

$$u(x) = \int_{R_n} \Gamma(x,\xi)\, X(\xi)\, d\xi \tag{D42}$$

<u>Theorem:</u> If M_1 and M_2 are linear differential matrix operators such that

$$M_1 M_2 = M_2 M_1 = 0 \tag{D43a}$$

and
$$M_1 - M_2 = \theta \cdot \mathbb{1} \tag{D43b}$$

where $\mathbb{1}$ = n-dimensional unit matrix = $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ (D43c)

θ = a scalar linear differential operator, (D43d)

then the fundamental solution $\Gamma(x,\xi)$ of (D41) is given by

$$\Gamma(x,\xi) = \frac{1}{k^2}\left\{ M_1\, \gamma_{\sqrt{\frac{k^2}{a}}} - M_2\, \gamma_{\sqrt{\frac{k^2}{b}}} \right\} \cdot \mathbb{1} \tag{D44}$$

where $\gamma_k(x,\xi)$ is the fundamental solu... on of the scalar differential equation formed with θ,

$$(\theta - k^2)\, v = - Y(x) \tag{D45}$$

Here $\gamma_k(x,\xi)$ is defined, in the usual way, by the requirement that $v(x)$ satisfying (D45) and homogeneous boundary conditions be given by

$$v(x) = \int_{R_n} \gamma_k(x,\xi)\, Y(\xi)\, d\xi \tag{D46}$$

The theorem will now be proved. In the proof it is convenient to use the fact that

$$\theta \gamma_0 = 0 \tag{D47}$$

in order to represent Γ in another form. We may write

$$\Gamma(x,\xi) = \left\{ \frac{1}{a} M_1 U_1 - \frac{1}{b} M_2 U_2 \right\} \cdot \mathbb{1} \tag{D48a}$$

when we define

$$U_1 = \frac{a}{k^2}\left\{ \gamma_{\sqrt{\frac{k^2}{a}}} - \gamma_0 \right\} \tag{D48b}$$

$$U_2 = \frac{b}{k^2}\left\{ \gamma_{\sqrt{\frac{k^2}{b}}} - \gamma_0 \right\} \tag{D48c}$$

This is equivalent to (D44). U_1 and U_2 are now regular at $x = \xi$ because the singularity of γ_k is independent of k.

Proof: The theorem will be proved if it is shown that

$$(a\, M_1 - b M_2 - k^2) \int_{R_n} \Gamma(x,\xi)\, X(\xi)\, d\xi = -X(x) \tag{D49}$$

which is equivalent to (D42). Using the second form of Γ (D48) we may transform the equation to be proved (D49) by taking the operators M_1 and M_2 in Γ outside the integral sign. This is allowed because U_1 and U_2 are regular for all x. Thus,

$$\left(M_1^2 - \frac{k^2}{a} M_1\right) \int_{R_n} U_1(x,\xi)\, X(\xi)\, d\xi - \left(-M_2^2 - \frac{k^2}{b} M_2\right) \int_{R_n} U_2(x,\xi)\, X(\xi)\, d\xi = -X \tag{D49'}$$

where we have used (D43). It also follows from (D43) that

$$M_1^2 = M_1 \mathcal{O} \cdot \mathcal{U}, \quad M_2^2 = -M_2 \mathcal{O} \cdot \mathcal{U}$$

so that the left hand side of (D49') may be transformed as follows:

$$M_1\left(\mathcal{O} - \frac{k^2}{a}\right) \int_{R_n} U_1 X\, d\xi - M_2\left(\mathcal{O} - \frac{k^2}{b}\right) \int_{R_n} U_2 X\, d\xi =$$

$$\frac{a}{k^2} M_1 \left\{ \left(\mathcal{O} - \frac{k^2}{a}\right) \int_{R_n} \gamma_{\sqrt{\frac{k^2}{a}}} X\, d\xi - \mathcal{O} \int_{R_n} \gamma_0 X\, d\xi \right\} + M_1 \int_{R_n} \gamma_0 X\, d\xi$$

$$- \frac{b}{k^2} M_2 \left\{ \left(\mathcal{O} - \frac{k^2}{b}\right) \int_{R_n} \gamma_{\sqrt{\frac{k^2}{b}}} X\, d\xi - \mathcal{O} \int_{R_n} \gamma_0 X\, d\xi \right\} - M_2 \int_{R_n} \gamma_0 X\, d\xi$$

$$= \frac{a}{k^2} M_1 \left\{ -X + X \right\} - \frac{b}{k^2} M_2 \left\{ -X + X \right\} + (M_1 - M_2) \int_{R_n} \gamma_0 X\, d\xi$$

Thus

$$(M_1 - M_2) \int_{R_n} \gamma_0 X\, d\xi \equiv \mathcal{O} \int_{R_n} \gamma_0 X\, d\xi = -X \tag{D50}$$

which proves (D49) and hence the theorem.

Note. The first form of Γ (D44) shows a splitting of Γ into two components $\Gamma^{(1)}$ and $\Gamma^{(2)}$, defined by

$$\Gamma^{(1)} = \frac{1}{k^2} M_1 \gamma_{\sqrt{\frac{k^2}{a}}} \quad , \quad \Gamma^{(2)} = -\frac{1}{k^2} M_2 \gamma_{\sqrt{\frac{k^2}{b}}} \tag{D51}$$

This splitting has the following properties for $\Gamma(x,y)$ considered as a function of x :

$$M_2 \Gamma^{(1)} = 0 \quad , M_1 \Gamma^{(2)} = 0 \tag{D52a}$$

$$(a M_1 - k^2) \Gamma^{(1)} = 0 \quad , \quad (-b M_2 - k^2) \Gamma^{(2)} = 0 \tag{D52b}$$

except at $x = \xi$

In the later applications of the general theorem proved here, this splitting will be shown to correspond to the splitting into longitudinal and transversal waves (§1.5).

De. General Non-stationary Waves

The theorem of (Dd) is now applied to find the fundamental solution Γ, S for the complete system of linearized equations for non-stationary flow in two and three space dimensions. These equations, written in terms of dimensionless variables and functions, are (cf. 1.37, 1.38):

$$T_1 q - \frac{3}{4} T_2 q - q_t - \operatorname{grad} s = X(x_i, t) \tag{D53a}$$

$$\operatorname{div} q + s_t = 0 \tag{D53b}$$

where q, and X are vector functions. In three space dimensions the operators T_1 and T_2 may be defined either by matrices or by standard vector operators:

$$T_2 = \text{curl curl} = \begin{pmatrix} -\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} & \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{\partial^2}{\partial x_1 \partial x_3} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & -\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_2 \partial x_3} \\ \frac{\partial^2}{\partial x_3 \partial x_1} & \frac{\partial^2}{\partial x_3 \partial x_2} & -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \end{pmatrix} \tag{D54a}$$

$$T_1 = \text{grad div} = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{\partial^2}{\partial x_1 \partial x_3} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_2 \partial x_3} \\ \frac{\partial^2}{\partial x_3 \partial x_1} & \frac{\partial^2}{\partial x_3 \partial x_2} & \frac{\partial^2}{\partial x_3^2} \end{pmatrix} \tag{D54b}$$

In two space dimensions, the corresponding matrix definition may be used.

It is not possible to apply the theorem of Dd directly because (D53) is a system of equations and because some first derivatives occur in these equations. However, if Laplace transforms are introduced, the first derivatives are eliminated and the system is easily transformed into the required form. We obtain, under zero initial conditions, the transformed system

$$T_1 \bar{q} - \frac{3}{4} T_2 \bar{q} - \sigma \bar{q} - \operatorname{grad} \bar{s} = -\bar{X}(x_i;\sigma) \tag{D55a}$$

$$\operatorname{div} \bar{q} + \sigma \bar{s} = 0 \tag{D55b}$$

The elimination of $\bar{s}$ gives

$$\left(\frac{1+\sigma}{\sigma}\right) T_1 \bar{q} - \frac{3}{4} T_2 \bar{q} - \sigma \bar{q} = -\bar{X}(x_i;\sigma) \tag{D56}$$

Equation (D56) is of the form (D41) and the operators T_1, T_2 have the required properties (D43):

$$T_1 T_2 = T_2 T_1 = 0 \tag{D57a}$$

$$T_1 - T_2 = \text{grad div} - \text{curl curl} = \Delta \cdot \mathcal{U} \tag{D57b}$$

An application of the theorem of Dd will enable us to find the fundamental solution of the transformed equation (D56). But by the reasoning of Da the fundamental solution of the transformed equation is just the transform $\bar{\Gamma}$ of the fundamental solution of the original system (D53). Thus, after an inversion, the fundamental solution will be obtained.

The theorem in Dd is now applied with the following table of correspondence:

$$\begin{array}{ll} M_i \rightarrow T_i & b \rightarrow \frac{3}{4} \\ \mathcal{Q} \rightarrow \Delta & k^2 \rightarrow \sigma \\ \multicolumn{2}{c}{a \rightarrow \frac{1+\sigma}{\sigma}} \end{array} \tag{D58}$$

Thus, γ_k are fundamental solutions for the differential operator $(\Delta - k^2)$, and are (with the customary definitions of r) in three space dimensions

$$\gamma_k = \frac{1}{4\pi r} e^{-kr} \tag{D59a}$$

and in two space dimensions

$$\gamma_k = \frac{1}{2\pi} K_o(kr) \tag{D59b}$$

Three-dimensional case

From (D44) and (D58) it follows that

$$\bar{\Gamma}(r;\sigma) = \frac{1}{\sigma}\left\{ T_1 \frac{e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}}{4\pi r} - T_2 \frac{e^{-r\sqrt{\frac{4\sigma}{3}}}}{4\pi r} \right\} \cdot \mathcal{U} \tag{D60}$$

in three space dimensions.

$\Gamma(r,t-\tau)$ is now obtained by the complex inversion formula

$$\Gamma(r,t-\tau) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left\{ \frac{1}{\sigma} T_1 \frac{e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}}{4\pi r} - \frac{1}{\sigma} T_2 \frac{e^{-r\sqrt{\frac{4\sigma}{3}}}}{4\pi r} \right\} e^{\sigma(t-\tau)} \, d\sigma \cdot \mathcal{U} \tag{D61}$$

The differential operators commute with the inversion integral so that

$$\Gamma(r,t-\tau) = T_1 \frac{1}{4\pi r}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)} \frac{e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}}{\sigma} d\sigma \cdot \mathcal{U} - T_2 \frac{1}{4\pi r}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)} \frac{e^{-r\sqrt{\frac{4\sigma}{3}}}}{\sigma} d\sigma \cdot \mathcal{U} \quad \text{(D6}$$

The first inversion integral is exactly that treated in Appendix A, (A9a), while the second is given in (Ref. 57). Denoting inverse Laplace transform by $\mathcal{L}^{-1}$ we have

$$\mathcal{L}^{-1}\left(\frac{1}{\sigma} e^{-r\sqrt{\frac{4\sigma}{3}}} \right) = \operatorname{erfc} \frac{r}{\sqrt{3t}} \tag{D63a}$$

$$\mathcal{L}^{-1}\left(\frac{1}{\sigma} e^{-r\frac{\sigma}{\sqrt{1+\sigma}}} \right) = \left[\frac{u(r,t)}{u_o} \right]_A \tag{D63b}$$

Hence

$$\Gamma(r,t-\tau) = T_1 \frac{1}{4\pi r}\left[\frac{u(r,t-\tau)}{u_o}\right]_A \cdot \mathcal{U} - T_2 \frac{1}{4\pi r}\,\mathrm{erfc}\,\frac{r}{\sqrt{3(t-\tau)}} \cdot \mathcal{U} \qquad \text{(D64)}$$

By using (D55) and (D60) we also obtain

$$\bar{S}(r;\sigma) = -\,\mathrm{div}\,\frac{1}{4\pi r}\,\frac{e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}}{1+\sigma}\cdot \mathcal{U} \qquad \text{(D65a)}$$

$$= -\,\mathrm{grad}\,\frac{1}{4\pi r}\,\frac{e^{-\frac{\sigma r}{\sqrt{1+\sigma}}}}{1+\sigma} \qquad \text{(D65b)}$$

The inversion of (D65) is also related to (D62) so that

$$S(r,t-\tau) = \mathrm{grad}\,\frac{1}{4\pi r}\,\frac{\partial}{\partial r}\left[\frac{u(r,t-\tau)}{u_o}\right]_A \qquad \text{(D66)}$$

<u>Note on splitting.</u>

From the remark about splitting made at the end of (Dd) it follows that the first part of $\bar{\Gamma}$ (denoted by $\bar{\Gamma}^{(1)}$) satisfies the equation

$$\left(1+\frac{1}{\sigma}\right)T_1\,\bar{\Gamma}^{(1)} - \sigma\,\bar{\Gamma}^{(1)} = 0 \qquad \text{(D67a)}$$

Furthermore, since $\mathrm{div}\,\Gamma^{(2)} = 0$, it follows from (D55b) that

$$\mathrm{div}\,\bar{\Gamma}^{(1)} + \sigma\,\bar{S} = 0 \qquad \text{(D67b)}$$

Hence, (D67a) becomes

$$T_1\,\bar{\Gamma}^{(1)} - \sigma\,\bar{\Gamma}^{(1)} - \mathrm{grad}\,\bar{S} = 0 \qquad \text{(D68)}$$

Applying $\mathcal{L}^{-1}$ to (D67) and putting

$$\Gamma^{(1)} \equiv \mathcal{L}^{-1}\bar{\Gamma}^{(1)} = T_1\left\{\frac{1}{4\pi r}\left[\frac{u(r,t-\tau)}{u_o}\right]_A\right\}\cdot \mathcal{U} \qquad \text{(D69)}$$

one obtains

$$T_1\,\Gamma^{(1)} - \Gamma^{(1)}_t - \mathrm{grad}\,S = 0 \qquad \text{(D70a)}$$

$$\mathrm{div}\,\Gamma^{(1)} + S_t = 0 \qquad \text{(D70b)}$$

$$\mathrm{curl}\,\Gamma^{(1)} = 0 \qquad \text{(D70c)}$$

except at $x_i = \xi_i$ i.e. $r = 0$

The last equation follows immediately from the definition of T_1.
Defining also

$$\Gamma^{(2)} \equiv \mathcal{L}^{-1}\,\bar{\Gamma}^{(2)} = T_2\left\{\frac{1}{4\pi r}\,\mathrm{erfc}\,\frac{r}{3\sqrt{t-\tau}}\right\}\cdot \mathbb{1} \tag{D71}$$

it easily follows that

$$-\frac{3}{4}\,T_2\,\Gamma^{(2)} - \Gamma_t^{(2)} = 0 \tag{D72a}$$

$$\mathrm{div}\,\Gamma^{(2)} = 0 \tag{D72b}$$

except at $r=0$

Hence the expression for Γ as given by (D64) actually represents the fundamental field as the sum of a longitudinal and a transversal flow field as defined in §1.5.

Two-dimensional case. The analysis is carried out for the two-dimensional case in the same way as for the three-dimensional case. Applying the theorem of (Dd), specifically (D44), and using (D58),

$$\bar{\Gamma}(r;\sigma) = \frac{1}{2\pi\sigma}\left\{T_1\,K_0\left(\frac{\sigma r}{\sqrt{\sigma+1}}\right) - T_2\,K_0\left(r\sqrt{\frac{4\sigma}{3}}\right)\right\}\cdot \mathbb{1} \tag{D73}$$

The inversion of the second term may be carried out as follows:

Let

$$f(r,t) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma t}\,\frac{K_0\left(r\sqrt{\frac{4\sigma}{3}}\right)}{\sigma}\,d\sigma \tag{D74}$$

Then

$$\frac{\partial f}{\partial r} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\sigma t}\sqrt{\frac{4}{3}}\,\frac{K_1\left(r\sqrt{\frac{4\sigma}{3}}\right)}{\sqrt{\sigma}}\,d\sigma$$

$$= -\sqrt{\frac{4}{3}}\left[\frac{1}{r}\sqrt{\frac{3}{4}}\right]e^{-\frac{r^2}{3t}} \qquad \text{from (Ref. 57)}$$

$$f(r,t) = \int_r^{\infty} e^{-\frac{\alpha^2}{3t}}\,\frac{d\alpha}{\alpha} \tag{D75}$$

Thus, using the complex inversion formula, the fundamental solution is

$$\Gamma(r,t-\tau) = T_1 \cdot \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\sigma(t-\tau)} \frac{1}{2\pi\sigma} K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right) d\sigma - T_2 \cdot \frac{1}{2\pi} \int_r^{\infty} \frac{1}{\alpha} e^{-\frac{\alpha^2}{3(t-\tau)}} d\alpha \tag{D76}$$

As before, we find S

$$S(r,t-\tau) = -\operatorname{grad} \mathcal{L}^{-1}\left[\frac{1}{2\pi\sqrt{1+\sigma}} K_o\left(\frac{\sigma r}{\sqrt{1+\sigma}}\right)\right] \tag{D77}$$

As in the three-dimensional case it is easily seen that the flow field is split, by the representation above, into a longitudinal and a transversal field.

Df. Stationary Waves in Two Dimensions

In this section the fundamental solutions will be found for the following set of equations of a two-dimensional stationary flow (cf. 1.43).

$$\left(\frac{4}{3}\nu T_1 - \nu T_2 - U\frac{\partial}{\partial x}\right)\vec{q} - c^2 \operatorname{grad} s = -\vec{X}(x,y) \tag{D78a}$$

$$\operatorname{div}\vec{q} + U\frac{\partial s}{\partial x} = 0 \tag{D78b}$$

where

$$T_1 = \text{grad div} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \tag{D78c}$$

$$T_2 = \text{curl curl} = \begin{pmatrix} -\frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial x^2} \end{pmatrix} \tag{D78d}$$

$\vec{q}$ is the velocity vector

The equations are given here in the dimensional form in order to show more clearly the dependence on the parameters ν, c, M, particularly the passage to the limit $c \to \infty$.

There are several methods available for obtaining the fundamental solutions of (D78). They could be obtained by integrating the corresponding non-stationary solutions, as described in §2.5 (method of descent). However, the resulting integrals are complicated and difficult to evaluate. For this reason it is more convenient to find the fundamental solutions directly, using the general method of (Dd).

In applying the general method it is necessary to eliminate one of the independent variables by means of transforms, as before (cf. De). In this case Fourier transforms are most convenient to use since the solution must be defined in the entire (x,y) plane. The introduction of the transforms into (D78) immediately reduces the system to a form in which the general theorem of (Dd) can be applied.

In this section we denote the Fourier transform of a function $f(x,y)$ by a wavy bar $\tilde{f}(\beta;y)$:

$$\tilde{f}(\beta;y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\beta x} f(x,y)\, dx \tag{D79a}$$

so that

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+i\beta x} \tilde{f}(\beta;y)\, d\beta \tag{D79b}$$

In transforming (D78) we make repeated use of the following relationship

$$\int_{-\infty}^{\infty} e^{-i\beta x} \frac{\partial u}{\partial x}\, dx = e^{-i\beta x} u(x,y) \Big|_{-\infty}^{+\infty} + i\beta \int_{-\infty}^{\infty} e^{-i\beta x} u(x,y)\, dx$$

$$= i\beta\, \tilde{u}(\beta;y) \tag{D80}$$

It is valid because $u(\pm\infty, y)$ is assumed to vanish.

Thus transformed operators $\tilde{T}_1$, $\tilde{T}_2$ are defined by replacing $\frac{\partial}{\partial x}$ by $i\beta$, and the transformed functions are introduced in (D78) to give

$$\left(\frac{4}{3}\nu \tilde{T}_1 - \nu \tilde{T}_2 - i\beta U\right)\tilde{\vec{q}} - c^2\, \widetilde{grad}\; \tilde{s} = -\tilde{\vec{X}}(\beta;y) \tag{D81a}$$

$$\widetilde{div}\; \tilde{u} + i\beta U \tilde{s} = 0 \tag{D81b}$$

Upon eliminating $\tilde{s}$

$$\left[\left(\frac{4}{3}\nu + \frac{c^2}{i\beta U}\right)\tilde{T}_1 - \nu\tilde{T}_2 - i\beta U\right]\tilde{\vec{q}} = -\tilde{\vec{X}} \tag{D82}$$

It is necessary to verify that the fundamental solution of (D81) is actually the Fourier transform of the fundamental solution Γ of (D78). This follows immediately from the definition of fundamental solution for the original and transformed equations and from the convolution theorem. The fundamental solution $\Gamma(x-\xi, y-\eta)$ is defined by

$$\vec{q}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x-\xi, y-\eta)\, \vec{X}(\xi,\eta)\, d\xi\, d\eta \tag{D83}$$

But transforming (D83) and using the convolution theorem (Ref. 58)

$$\tilde{\vec{q}}(\beta; y) = \sqrt{2\pi} \int_{-\infty}^{\infty} \tilde{\Gamma}(\beta; y-\eta)\, \tilde{\vec{X}}(\beta,\eta)\, d\eta \tag{D84}$$

Thus $\sqrt{2\pi}\,\tilde{\Gamma}$ is the fundamental solution of (D82) for $\tilde{\vec{q}}$. The same relationship is true for the fundamental solution for the condensation s

$$s(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x-\xi, y-\eta)\, \vec{X}(\xi,\eta)\, d\xi\, d\eta \tag{D85a}$$

$$\tilde{s}(\beta,y) = \sqrt{2\pi} \int_{-\infty}^{\infty} \tilde{S}(\beta; y-\eta)\, \tilde{\vec{X}}(\beta;\eta)\, d\eta \tag{D85b}$$

The following relationship between the modified operators is valid after the transformation

$$\tilde{T}_1 \tilde{T}_2 = \tilde{T}_2 \tilde{T}_1 = 0 \tag{D86a}$$

$$\tilde{T}_1 - \tilde{T}_2 = \tilde{\Delta} \cdot \mathcal{U} \tag{D86b}$$

Hence the general theorem of (Dd) applies immediately with the following table of correspondence

$$M_1 \to \tilde{T}_1 \qquad M_2 \to \tilde{T}_2$$

$$\theta \to \tilde{\Delta} \qquad a \to \frac{4}{3}\nu + \frac{c^2}{i\beta U}$$

$$b \to \nu \qquad k^2 \to i\beta U \tag{D87}$$

The fundamental solution γ of

$$(\Theta - k^2) f = (\tilde{\Delta} - k^2) f = \frac{\partial^2}{\partial y^2} - (k^2+\beta^2) f = 0 \tag{D88}$$

is (for $\eta = 0$)

$$\gamma_k = \frac{1}{2\sqrt{k^2+\beta^2}} e^{-|y|\sqrt{\beta^2+k^2}} , \quad \mathcal{R}\ell \sqrt{\beta^2+k^2} > 0 \tag{D89}$$

Hence, by (D48)

$$\sqrt{2\pi}\,\tilde{\Gamma}(\beta, y) = \frac{1}{k^2}\left(\tilde{T}_1 \gamma_{\sqrt{\frac{k^2}{a}}} - \tilde{T}_2 \gamma_{\sqrt{\frac{k^2}{b}}}\right) \cdot \mathcal{U}$$

$$= \frac{1}{k^2} \quad \tilde{T}_1 \frac{1}{2\sqrt{\frac{k^2}{a}+\beta^2}} e^{-|y|\sqrt{\frac{k^2}{a}+\beta^2}} - \tilde{T}_2 \frac{1}{2\sqrt{\frac{k^2}{b}+\beta^2}} e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}} \tag{D90}$$

$\tilde{\Gamma}$ is represented here as the difference of a longitudinal wave and a transversal wave, just as for the non-stationary waves. Now, as $c \to \infty$, (incompressible case) the second term remains unchanged since b does not depend on c. The first term tends to $\frac{1}{k^2}\tilde{T}_1 \gamma_0$ which is equal to $\frac{1}{k^2}\tilde{T}_2 \gamma_0$, since $\tilde{\Delta}\gamma_0 = 0$ by definition. Then

$$\tilde{\Gamma}_0 = \lim_{c\to\infty} \tilde{\Gamma} = \frac{1}{\sqrt{2\pi k^2}} \tilde{T}_2 \left(\gamma_0 - \gamma_{\sqrt{\frac{k^2}{b}}}\right)$$

$$= \frac{1}{\sqrt{2\pi k^2}} \tilde{T}_2 \left\{ \frac{1}{2|\beta|} e^{-|y||\beta|} - \frac{1}{2\sqrt{\frac{k^2}{b}+\beta^2}} e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}} \right\} \tag{D91}$$

and the fundamental solution for the incompressible case is obtained by inversion as

$$\Gamma_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\beta x}\, \tilde{\Gamma}_0(\beta; y)\, d\beta \tag{D92}$$

Now

$$\frac{1}{k^2}\tilde{T}_2 = \frac{1}{i\beta U}\begin{pmatrix} -\frac{\partial^2}{\partial y^2} & i\beta\frac{\partial}{\partial y} \\ i\beta\frac{\partial}{\partial y} & -\beta^2 \end{pmatrix} = \frac{1}{U}\begin{pmatrix} \frac{-1}{i\beta}\frac{\partial^2}{\partial y^2} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix}$$

Thus putting $\Gamma_o = (\Gamma_{o_{ij}})$ **we see that**

$$\Gamma_{o_{12}} = \Gamma_{o_{21}} = \frac{1}{2\pi U}\frac{\partial}{\partial y}\int_{-\infty}^{\infty} e^{i\beta x}\left[\frac{e^{-|y||\beta|}}{2|\beta|} - \frac{1}{2\sqrt{\frac{k^2}{b}+\beta^2}}\, e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}}\right] d\beta \qquad \text{(D93a)}$$

$$\Gamma_{o_{22}} = \frac{-1}{2\pi U}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} e^{i\beta x}\left[\frac{e^{-|y||\beta|}}{2|\beta|} - \frac{1}{2\sqrt{\frac{k^2}{b}+\beta^2}}\, e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}}\right] d\beta \qquad \text{(D93b)}$$

or defining

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\beta x}\ \frac{e^{-|y||\beta|}}{|\beta|}\, d\beta \qquad \text{(D94a)}$$

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\beta x}\ \frac{e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}}}{\sqrt{\frac{k^2}{b}+\beta^2}}\, d\beta \qquad \text{(D94b)}$$

we have

$$\Gamma_{o_{21}} = \Gamma_{o_{12}} = \frac{1}{2\pi U}\frac{\partial}{\partial y}(I_1 - I_2) \qquad \text{(D95a)}$$

$$\Gamma_{o_{22}} = -\frac{1}{2\pi U}\frac{\partial}{\partial x}(I_1 - I_2) \qquad \text{(D95b)}$$

Further

$$\begin{aligned}\frac{1}{i\beta U}\frac{\partial^2}{\partial y^2}\left\{\gamma_o - \gamma_{\sqrt{\frac{k^2}{b}}}\right\} &= \frac{1}{i\beta U}\left\{\frac{|\beta|}{2}\, e^{-|y||\beta|} - \frac{\sqrt{\frac{k^2}{b}+\beta^2}}{2}\, e^{-|y|\sqrt{\frac{k^2}{b}+\beta^2}}\right\}\\ &= \frac{1}{i\beta U}\left\{\frac{|\beta|}{2}\, e^{-|y||\beta|} - \frac{\beta^2 + \frac{i\beta U}{\nu}}{2\sqrt{\frac{i\beta U}{\nu}+\beta^2}}\, e^{-|y|\sqrt{\frac{i\beta U}{\nu}+\beta^2}}\right\}\\ &= \frac{-1}{\nu}\gamma_{\sqrt{\frac{k^2}{b}}} + \frac{i\beta}{U}\left\{\gamma_{\sqrt{\frac{k^2}{b}}} - \gamma_o\right\}\end{aligned}$$

From this it follows that

$$\Gamma_{o_{11}} = +\frac{1}{2\pi U}\frac{\partial}{\partial x}(I_1 - I_2) + \frac{1}{2\pi\nu} I_2 \qquad \text{(D96)}$$

Thus the evaluation of the incompressible tensor depends on the evaluation of I_1 and I_2 and this may be done explicitly as follows:

$$\frac{\partial I_1}{\partial y} = -\frac{1}{2}\,\mathrm{sign}\;y \int_{-\infty}^{\infty} e^{i\beta x - |y||\beta|}\,d\beta = \mathrm{sign}\;y \int_{0}^{\infty} e^{-|y|\beta} \cos\beta x\,d\beta = \frac{(-\mathrm{sign}\;y)|y|}{x^2+y^2}$$

$$= -\frac{y}{x^2+y^2}$$

$$I_1 = -\log r \tag{D97a}$$

and

$$I_2 = \frac{1}{2}\int_{-\infty}^{\infty} e^{i\beta x}\,\frac{e^{-|y|\sqrt{\beta^2+\frac{i\beta U}{\nu}}}}{\sqrt{\beta^2+\frac{i\beta U}{\nu}}}\,d\beta \tag{D97b}$$

Considered as an integration in the complex β plane the path of integration should be taken as on Fig. D.2. By choosing the correct sheet of the Riemann surface we have $\mathcal{R}\ell\sqrt{\beta^2+\frac{i\beta U}{\nu}} > 0$ on the contour. The integral is transformed to a known form if the contour is shifted to $\mathcal{I}m\,\beta = -\frac{U}{2\nu}$, and of course the integrals on the contours are equivalent.

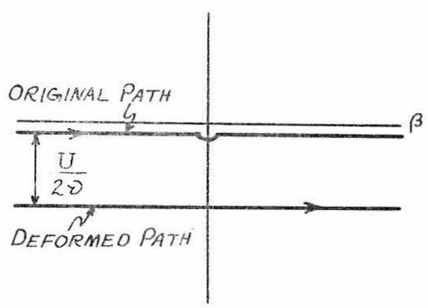

Figure D.2

Thus

$$I_2 = \frac{1}{2} e^{\frac{Ux}{2\nu}} \int_{-\infty}^{\infty} e^{i\beta x} \frac{e^{-\sqrt{\beta_1^2 + \frac{U^2}{4\nu^2}}\,|y|}}{\sqrt{\beta_1^2 + \frac{U^2}{4\nu^2}}}$$

This integral is tabulated (Ref. 57) as

$$I_2 = e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) \tag{D97b'}$$

Thus we can write the fundamental solution:

$$\Gamma_0 = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \end{pmatrix} I \cdot \mathbb{1} + \frac{1}{2\pi U} e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{D98a}$$

where

$$I = \frac{1}{2\pi U}(I_1 - I_2) = -\frac{1}{2\pi U}\left[\log r + e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right)\right] \tag{D98b}$$

In the compressible case a correction tensor has to be added to Γ_0. We have

$$\Gamma = \Gamma_0 + \Gamma_c$$

with

$$\Gamma_c = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\beta x}}{i\beta U} \begin{pmatrix} -\beta^2 & i\beta \frac{\partial}{\partial y} \\ i\beta \frac{\partial}{\partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \left(\gamma_{\sqrt{\frac{k^2}{a}}} - \gamma_0 \right) d\beta \tag{D99}$$

Now

$$\gamma_{\sqrt{\frac{k^2}{a}}} = \frac{1}{2\sqrt{\frac{k^2}{a} + \beta^2}} e^{-|y|\sqrt{\frac{k^2}{a}+\beta^2}} = \frac{1}{2\sqrt{\frac{k^2}{\frac{4}{3}\nu + \frac{c^2}{i\beta U}} + \beta^2}} e^{-|y|\sqrt{\frac{k^2}{\frac{4}{3}\nu + \frac{c^2}{i\beta U}} + \beta^2}}$$

$$\gamma_{\sqrt{\frac{k^2}{a}}} = \frac{1}{2}\sqrt{\frac{1 + R^2 M^2 \beta^2}{\beta^2(1-M^2) + R^2 M^2 \beta^4 + i(RM^3\beta^3)}}\ \exp\left[-|y|\sqrt{\frac{\beta^2(1-M^2) + R^2M^2\beta^4 + i(RM^3\beta^3)}{1 + R^2M^2\beta^2}}\right] \tag{D100}$$

where $\mathcal{R}\ell\sqrt{} > 0$ on the contour and where $R = \frac{4}{3}\frac{\nu}{c}$, $M = \frac{U}{c}$.

Asymptotic Estimates of Velocity Field: Subsonic case. It is easy to obtain a crude formula in the subsonic case $(M<1)$. By considering only contributions for small β, an asymptotic formula for large y can be found and the error could be estimated (cf. Appendix A). Here only the first term will be obtained.

Thus we approximate

$$\frac{\gamma}{\sqrt{\frac{k^2}{a}}} \sim \frac{1}{2\beta\sqrt{1-M^2}} \exp\left[-|y||\beta|\sqrt{1-M^2}\right] \tag{D101}$$

It is clear that this is a valid approximation also if $R = \frac{4}{3}\frac{\nu}{c}$ is small, and is actually the value approached by $\frac{\gamma}{\sqrt{\frac{k^2}{a}}}$ as $R \to 0$. Thus

$$\Gamma_{c_{11}} \sim \frac{1}{4\pi U}\int_{-\infty}^{\infty} e^{i\beta x}\,\text{sign}\,\beta\,\frac{e^{-|y||\beta|\sqrt{1-M^2}}}{\sqrt{1-M^2}}\,d\beta - \frac{1}{4\pi U}\int_{-\infty}^{\infty} e^{i\beta x}\,e^{-|y||\beta|}\,\text{sign}\,\beta\, d\beta$$

$$\sim -\frac{1}{2\pi U}\int_0^{\infty} \frac{\text{sign}\,\beta\, x\, e^{-|y|\sqrt{1-M^2}\,\beta}}{\sqrt{1-M^2}}\,d\beta$$

$$\sim -\frac{1}{2\pi U}\left\{\frac{x/\sqrt{1-M^2}}{x^2+(1-M^2)y^2} - \frac{x}{x^2+y^2}\right\}$$

If we now write out (D98)

$$\Gamma_{o_{11}} = -\frac{1}{2\pi U}\left\{\frac{x}{r^2} + \frac{U}{2\nu}e^{\frac{Ux}{2\nu}}K_o\left(\frac{Ur}{2\nu}\right) - e^{\frac{Ux}{\nu}}\frac{U}{2\nu}K_1\left(\frac{Ur}{2\nu}\right)\frac{x}{r}\right\} + \frac{1}{2\pi U}e^{\frac{Ux}{2\nu}}K_o\left(\frac{Ur}{2\nu}\right)$$

Adding the correction term $\Gamma_{c_{11}}$, we have the approximate result for large y $(M<1)$

$$\Gamma_{11} = \Gamma_{o_{11}} + \Gamma_{c_{11}} \sim -\frac{1}{2\pi U}\left\{\frac{x/\sqrt{1-M^2}}{x^2+(1-M^2)y} - \frac{U}{\nu}e^{\frac{Ux}{2\nu}}K_o\left(\frac{Ur}{2\nu}\right) - \frac{U}{2\nu}\frac{x}{r}e^{\frac{Ux}{2\nu}}K_1\left(\frac{Ur}{2\nu}\right)\right\} \tag{D102}$$

We obtain, in the same way, the following formulae for the other correction components of the fundamental tensor

$$\Gamma_{c_{12}} = \Gamma_{c_{21}} \sim \frac{1}{2\pi U} \left\{ \frac{y}{x^2+y^2} - \frac{y\sqrt{1-M^2}}{x^2+(1-M^2)\,y^2} \right\} \tag{D103a}$$

$$\Gamma_{c_{22}} \sim \frac{1}{2\pi U} \left\{ \frac{x\sqrt{1-M^2}}{x^2+(1-M^2)\,y^2} - \frac{x}{x^2+y^2} \right\} \tag{D103b}$$

<u>Formulas for Pressure:</u> Using (D81b) and the formulas for $\tilde{\Gamma}$ we obtain

$$\tilde{S}(\beta;y) = -\frac{1}{i\beta U}\,\widetilde{div}\,\tilde{\Gamma} = -\frac{1}{i\beta U}\,\widetilde{div}(\tilde{\Gamma}_o + \tilde{\Gamma}_c) = -\frac{1}{i\beta U}\,\widetilde{div}\,\tilde{\Gamma}_c$$

or

$$\tilde{S} \equiv (\tilde{S}_1, \tilde{S}_2) = \frac{1}{\sqrt{2\pi}\,U^2}\left[-i\beta\left(1-\frac{1}{\beta^2}\frac{\partial^2}{\partial y^2}\right),\ -\frac{\partial}{\partial y}\left(1-\frac{1}{\beta^2}\frac{\partial^2}{\partial y^2}\right)\right]\gamma_{\sqrt{\frac{k^2}{a}}} \tag{D104}$$

But

$$\left(1-\frac{1}{\beta^2}\frac{\partial^2}{\partial y^2}\right)\gamma_{\sqrt{\frac{k^2}{a}}} = \left(1-\frac{|\beta|^2(1-M^2)}{\beta^2}\right)\gamma_{\sqrt{\frac{k^2}{a}}} = M^2\gamma_{\sqrt{\frac{k^2}{a}}} \tag{D105}$$

to the same degree of approximation as was used for (D101). Thus $\tilde{S}_1$ is merely a factor times the first part of $\Gamma_{c_{11}}$ or the potential part occurring in the solution for Γ_{11}

$$S_1 \sim \frac{1}{2\pi c^2}\,\frac{1}{\sqrt{1-M^2}}\left\{\frac{x}{x^2+y^2\,(1-M^2)}\right\} \tag{D106}$$

and as $c \to \infty$

$$c^2 S_1 \to \frac{1}{2\pi}\,\frac{x}{x^2+y^2} \tag{D107}$$

Similarly

$$S_2 \sim \frac{1}{2\pi c^2}\quad\frac{|y|\sqrt{1-M^2}}{x^2+(1-M^2)\,y^2} \tag{D108}$$

<u>Supersonic Case</u>

For $M>1$ such simple considerations do not even give a qualitative picture. They lead to divergent integrals which can be interpreted as impulse functions (derivatives of step functions). A better approximation is now given which indicates this in the limit $\partial \to 0$.

First a determination of the branches of the square root in $\gamma\sqrt{\frac{k^2}{a}}$ must be given. Write

$$\frac{\beta^2(1-M^2)+R^2M^2\beta^4+iRM^3\beta^3}{1+R^2M^2\beta^2}=\mathcal{R}e^{i\theta} \qquad \text{(D109a)}$$

with the following determination of θ :

$$\begin{array}{ll} \frac{m}{MR}\leq|\beta|\leq\infty & \theta=\alpha \\ 0\leq|\beta|\leq\frac{m}{MR} & \theta=\alpha+\pi \\ -\frac{m}{MR}\leq|\beta|\leq 0 & \theta=\alpha-\pi \end{array}$$

where:

$$\mathcal{R}=\sqrt{\left[\beta^2(1-M^2)+R^2M^2\beta^4\right]^2+R^2M^6\beta^6}\cdot\frac{1}{1+R^2M^2\beta^2}$$

$$\doteq|\beta|^2\,|1-M^2| \qquad \textbf{for small } \beta \qquad \text{(D109b)}$$

and

$$\alpha=\tan^{-1}\frac{RM^3\beta^3}{R^2M^2\beta^4-(M^2-1)\beta^2} \qquad m^2=M^2-1>0 \qquad \text{(D109c)}$$

$$\alpha\doteq-\frac{M^3R\beta}{m^2} \qquad |\alpha|\leq\frac{\pi}{2} \qquad \textbf{for small } \beta \qquad \text{(D109d)}$$

In this way $\pm\infty$ is on the same branch. With this determination we obtain the following exact formula from (D99) for the correction tensor:

$$\Gamma_c(x,y)=\frac{1}{2\pi U}\int_{-\infty}^{\frac{-m}{MR}}\frac{e^{i\beta x}}{i\beta}\tilde{T}_1\left\{\frac{e^{-|y|\mathcal{R}^{\frac{1}{2}}e^{i\frac{\alpha}{2}}}}{2\mathcal{R}^{\frac{1}{2}}e^{i\frac{\alpha}{2}}}-\frac{e^{-|y||\beta|}}{2|\beta|}\right\}d\beta$$

$$+\frac{1}{2\pi U}\int_{\frac{-m}{MR}}^{0}\frac{e^{i\beta x}}{i\beta}\tilde{T}_1\left\{\frac{e^{-|y|\mathcal{R}^{\frac{1}{2}}e^{i(\frac{\alpha}{2}-\frac{\pi}{2})}}}{2\mathcal{R}^{\frac{1}{2}}e^{i(\frac{\alpha}{2}-\frac{\pi}{2})}}-\frac{e^{-|y||\beta|}}{2|\beta|}\right\}d\beta$$

$$+\frac{1}{2\pi U}\int_{0}^{\frac{m}{MR}}\frac{e^{i\beta x}}{i\beta}\tilde{T}_1\left\{\frac{e^{-|y|\mathcal{R}^{\frac{1}{2}}e^{i(\frac{\alpha}{2}+\frac{\pi}{2})}}}{2\mathcal{R}^{\frac{1}{2}}e^{i(\frac{\alpha}{2}+\frac{\pi}{2})}}-\frac{e^{|y||\beta|}}{2|\beta|}\right\}d\beta$$

$$+\frac{1}{2\pi U}\int_{\frac{m}{MR}}^{\infty}\frac{e^{i\beta x}}{i\beta}\tilde{T}_1\left\{\frac{e^{-|y|\mathcal{R}^{\frac{1}{2}}e^{i\frac{\alpha}{2}}}}{2\mathcal{R}^{\frac{1}{2}}e^{i\frac{\alpha}{2}}}-\frac{e^{-|y||\beta|}}{2|\beta|}\right\}d\beta \qquad \text{(D110)}$$

An approximate formula is obtained as before by assuming that most of the contribution comes from $\int_{\frac{-m}{MR}}^{0}$ and $\int_{0}^{\frac{m}{MR}}$ while the other

integrals are neglected. It is further assumed that the former extend to ∞ ($\frac{m}{MR} \longrightarrow \infty$, good for small ϑ) and that only the small values of β are important. This approximation is of course not valid near the origin $(y=0, x=0)$.

Then, neglecting higher powers of β we introduce these approximations into (D109)

$$\mathcal{R}^{\frac{1}{2}} e^{i\left(\frac{\alpha}{2}-\frac{\pi}{2}\right)} \doteq |\beta| m \left\{\sin\frac{\alpha}{2} - i\cos\frac{\alpha}{2}\right\} \doteq |\beta| m \left\{\frac{\alpha}{2} - i\right\}$$

$$\mathcal{R}^{\frac{1}{2}} e^{i\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)} \doteq |\beta| m \left\{-\sin\frac{\alpha}{2} + i\cos\frac{\alpha}{2}\right\} \doteq |\beta| m \left\{-\frac{\alpha}{2} + i\right\}$$

where

$$\frac{\alpha}{2} = -\frac{M^3 \beta R}{2m^2}$$

Thus (D110) is approximated by

$$\Gamma_c(x,y) \cong \frac{1}{2\pi U}\int_{-\infty}^{0} \frac{e^{i\beta x}}{i\beta}\,\tilde{T}_1 \left\{\frac{i e^{-|y||\beta| m\left[-\frac{M^3\beta R}{2m^2} - i\right]}}{2|\beta| m} - \frac{e^{-|y||\beta|}}{2|\beta|}\right\} d\beta$$

$$+ \frac{1}{2\pi U}\int_{0}^{\infty} \frac{e^{i\beta x}}{i\beta}\,\tilde{T}_1 \left\{\frac{i e^{-|y||\beta| m\left[\frac{M^3\beta R}{2m^2} + i\right]}}{2|\beta| m} - \frac{e^{-|y||\beta|}}{2|\beta|}\right\} d\beta \tag{D111}$$

We are thus led to consider

$$\Gamma_{c_{11}} \cong \frac{1}{2\pi U}\int_{-\infty}^{0} e^{i\beta x}\, \frac{1}{2m}\, e^{-|y||\beta| m}\, e^{-\frac{|y| R M^3}{2m}\beta^2}\, d\beta - \frac{1}{2\pi U}\int_{-\infty}^{\infty} \frac{e^{i\beta x}}{i}\,\frac{e^{-|y||\beta|}}{2}\, d\beta$$

$$+ \frac{1}{2\pi U}\int_{0}^{\infty} \frac{e^{i\beta x}}{2m}\, e^{-|y||\beta| m i}\, e^{-\frac{|y| R M^3}{2m}\beta^2}\, d\beta + \frac{1}{2\pi U}\int_{0}^{\infty} \frac{e^{i\beta x}}{i}\,\frac{e^{-|y||\beta|}}{2}\, d\beta$$

$$\cong \frac{1}{2\pi U M}\int_{0}^{\infty} e^{-\frac{|y| R M^3}{2m}\beta^2} \cos\beta\,(x - m|y|)\, d\beta + \frac{1}{2\pi U}\int_{0}^{\infty} e^{-|y||\beta|} \sin\beta x\, d\beta$$

$$\cong \frac{1}{2\pi U}\left\{\frac{1}{M^{3/2}}\sqrt{\frac{\pi}{2m\,\frac{8}{3}\,\frac{\partial}{c}\,|y|}}\; e^{-\frac{m(x-m|y|)^2}{\frac{8}{3}\frac{\partial |y|}{c} M^3}} + \frac{x}{x^2+y^2}\right\} \tag{D112}$$

We notice that as $\vartheta \to 0$, (the first term in $\Gamma_{c_{11}}$) $\to 0$, except on the lines $x = m|y|$, where it becomes infinite. In a similar way we obtain

$$\Gamma_{c_{21}} = \Gamma_{c_{12}} \cong \frac{-\operatorname{sign}(y)}{2\pi U} \left\{ \frac{1}{M^{3/2}} \sqrt{\frac{\pi}{m\frac{8}{3}\frac{\vartheta}{c}|y|}}\; e^{\frac{-m(x-m|y|)^2}{\frac{8}{3}\frac{\vartheta}{c}|y|M^3}} - \frac{|y|}{x^2+y^2} \right\} \tag{D113}$$

In order to determine s we have again to consider (cf. (D104))

$$\left(1 - \frac{1}{\beta^2}\frac{\partial^2}{\partial y^2}\right)\gamma_{\sqrt{\frac{k^2}{a}}} = \left(1 - \frac{1}{\beta^2}\mathcal{R}e^{i\theta}\right)\gamma_{\sqrt{\frac{k^2}{a}}} \doteq -M^2\gamma_{\sqrt{\frac{k^2}{a}}} \qquad \text{as before.}$$

Thus there is the same relationship between s_1 and the irrotational part of Γ_{11} as before and

$$s_1 \cong -\frac{1}{2\pi c^2} M^{\frac{3}{2}} \sqrt{\frac{\pi}{m\frac{8}{3}\frac{\vartheta}{c}|y|}}\; e^{\frac{-m(x-m|y|)^2}{M^3\frac{8}{3}\frac{\vartheta}{c}|y|}} \tag{D114}$$

LIST OF SYMBOLS

Symbol	Meaning	Page where introduced
c	Adiabatic velocity of sound in the free stream	24
c_a	Adiabatic velocity of sound	20
c_i	Isothermal velocity of sound	20
c_p	Specific heat at constant pressure	15
c_v	Specific heat at constant volume	15
E	Internal energy	14
$\vec{F}$	External force per unit mass	14
$\mathcal{G}$	Piston curve	44
I	Entropy	18
$\vec{i}, \vec{j}, \vec{k}$	Unit vectors parallel to cartesian axes	25
k	Heat conduction coefficient	4
K_o, K_1	Modified Bessel functions of second kind	77
M	Free stream Mach number, U/c	34
M_1, M_2	Linear matrix operators	172
P	Pressure	14
p	Pressure perturbation	22
$\vec{Q}$	Velocity vector	14
$\vec{q}$	Velocity perturbation vector	22
$\mathcal{Q}$	Heat added to flow per unit mass	14
$\bar{R}$	Gas constant	15
r	Radius	42
s	Condensation	22
S	Fundamental solution for s	94

t	Time coordinate	14
t'	Dimensionless time coordinate	49
T	Temperature	14
T_1, T_2	Linear matrix operators	176
U	Free-stream velocity	22
u_i; u, v, w	Velocity perturbation components	15
x_i; x, y, z	Cartesian coordinates	15
x_i'; x', y', z'	Dimensionless space coordinates	49
$\vec{X}$	External force perturbation per unit mass	22
$\vec{Z}$	Singular force perturbation per unit mass	85
β	Variable of Fourier transformation	86
γ	Ratio of specific heats, c_p/c_v	4
γ_k	Fundamental solution	173, 178
Γ_{ij}	Fundamental vector solution tensor	84
δ	Dirac function	42
Δ	Laplacian operator $\Delta A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$	23
η	Parabolic coordinate	141
Θ	Temperature perturbation	22
θ	Scalar operator	173
λ	Basic viscosity coefficient	15
Ξ	Potential of external force per unit mass	29
μ	Basic viscosity coefficient	4
ν	Kinematic viscosity coefficient	24
$\bar{\nu} = \frac{8\nu^*}{3(\gamma+1)}$	Modified viscosity coefficient	152
ρ	Density	14
σ	Variable of Laplace transformation	49
Φ	Velocity potential	34
φ	Velocity perturbation potential	28

ψ	Stream function	70
χ	Dissipation function	14
$\vec{\omega}$	Vorticity	24
$(_0)$	Undisturbed flow conditions	22
$(_1)(_2)$	Longitudinal & transversal parts of field	32
$(\rightarrow)$	Vector	14
$(^-)$	Laplace transform	131
$(^\sim)$	Fourier transform	118, 183
$(^*)$	Sonic flow conditions	146
grad $\vec{A}$	Gradient: grad $\vec{A} = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$	24
div $\vec{A}$	Divergence: div $\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	24
curl $\vec{A}$	Operator defined by curl $\vec{A}$ = $\vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$	24
$\mathbb{1}$	Unit matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$	173
erfc	Complementary error function	49

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ERRATA AND ADDENDA

to

" Problems in the Theory of Viscous Compressible Fluids "

by P. A. Lagerstrom, J. D. Cole and L. Trilling

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PREFACE TO THE REPRINTED EDITION

With permission of the Office of Naval Research, the Durand Reprinting Committee, California Institute of Technology, has undertaken the reprinting and distribution of this report.

Pages i-iv and 1-200 are an exact reprint of the original edition. The following sections have been added:

These sections were prepared by J. D. Cole, G. E. Latta and P. A. Lagerstrom. Two papers, closely connected with this report are published separately: Reference 83 which complements some of the work of Appendix D, and Reference 84 which surveys some of the problems of the report.

ERRATA

Page	Line	Correction
11	11*	Replace (2.27a) by (2.28a)
14	Eq. (1.11)	This equation is correct only for constant μ. In general, in tensor notation, it should read $$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial P}{\partial x_i} - \frac{2}{3} \frac{\partial}{\partial x_i}\left(\mu \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$
16	Eq. (1.11')	Correct only for constant μ. In general, should read $$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$
18	Eq. (1.22)	Add 0 in bottom row, third column
23	10*	Delete "the left hand side of"
24	6	Should read: where $C = \sqrt{\gamma \frac{P_0}{\rho_0}}$
30	14	Replace (1.54d) by (1.54a)
37	16	Should read: a certain value, usually zero, of . . .
71	Eq. (2.61d)	Sign of ω should be changed here; also on p. 89, Eqs. (2.816) and (2.820), and p. 104, lines 13 and 14.
76	Eq. (2.613)	Replace $e^{-\frac{(x-Ut)^2 t^2}{4\nu t}}$ by $e^{-\frac{(x-Ut)^2+y^2}{4\nu t}}$

*Count from bottom of page

Page	Line	Correction
76	Eq. (2.613)	Replace $e^{\frac{Ux}{2\nu}} K_o\left(\frac{Ur}{2\nu}\right)$ by $2e^{\frac{Ux}{2\nu}} K_o\left(\frac{Ur}{2\nu}\right)$
76	Eq. (2.614)	Insert minus sign before second integral
		Replace $-\frac{Uy e^{\frac{Ux}{2\nu}}}{2\nu r} K_1\left(\frac{Ur}{2\nu}\right)$ by $-\frac{Uy e^{\frac{Ux}{2\nu}}}{\nu r} K_1\left(\frac{Ur}{2\nu}\right)$
90	2*	Replace (2.612a) by (2.621a)
92	14	Replace $S \to \infty$ by $c^2 \to \infty$
100	Eq. (2.852)	Should read: $K_o\left(\frac{\sigma r}{\sqrt{c^2 + \frac{4\nu\sigma}{s}}}\right)$
117	Eqs. (3.44a)	Equations are: $u_{1y} - v_{1x} = 0$
	(3.44b)	$u_{2x} + v_{2y} = 0$
131	Eq. (A6b)	Should read $\int_0^\infty e^{-\sigma t} \frac{u_o}{\delta}\, dt$
132	Eq. (A10a)	Replace $\mathcal{E}_o$ by $\mathcal{E}_o u_o$
	Eq. (A10b)	Replace $\mathcal{E}_1$ by $\mathcal{E}_1 u_o$
134	Fig. A1	Identation at origin should be clearly shown
136	5	Should read: $\mathcal{J}_2 = \int_D^E$ and let $\sigma = -1 + re^{i\pi}$
137	9*	Delete π in exponential in integral for $\mathcal{J}_4$
	1*	Delete $+\mathcal{J}_{4_1}$ at end of page
140	1	Replace $\omega^{a-\frac{2}{3}}$ by $\omega^{a-\frac{3}{2}}$
	2	Replace $\omega'^{2\,a-3}$ by $\omega'^{(2a-3)}$
141	Eq. (A22)	Replace $e^{i\omega b}$ by $e^{2i\omega b}$
142	Eq. (A30)	Replace t by τ in both places

Page	Line	Correction
143	Eq. (A32)	Insert brackets so that $\frac{u_o}{2\pi i}$ multiplies both integrals
145	Eq. (A36)	Replace $\mathcal{E}$ by $\mathcal{E} u_o$
148	Eq. (B8)	Should read: $p^*_{-\infty}(x,t) = \ldots C^*(W_{-\infty} - \varphi_x)\}$
	Eq. (B10)	Should read: $\frac{u}{\rho} P_x = \ldots - \left(\frac{rP}{\rho}\right) u_x - \frac{1}{\rho} P_t + \ldots$
	Eq. (B11)	Should read: $u u_t + \left(u^2 - \frac{rP}{\rho}\right) u_x = \frac{1}{\rho} P_t + \ldots$
	Eq. (B12)	Should read: $C^* \varphi_{xt} + \left\{2C^* \varphi_x - C^{*2}(p^* - s^*)\right\} \varphi_{xx} = \frac{C^{*2}}{\gamma} p^*_t + \frac{4}{3} \vartheta^* C^* \varphi_{xxx}$
149	Eq. (B13b)	Should read: $\frac{p^*}{\gamma} - s^* \Big\|_{-\infty}^{-C^*t} = \ldots$
	Eq. (B14)	Replace $\gamma + 1$ by $\gamma - 1$
	Eq. (B15)	Should read: $\ldots \left\{\frac{4}{3} \vartheta^* \varphi_{xx} - \varphi_t - C^* \varphi_x\right\} + J(x,t)$
	Eq. (B16)	Replace ϑ by ϑ^*
150	Eq. (B17a)	Replace $-C^* J(\varphi)$ by $-C^* \varphi_{xx} J(\varphi)$
155	Eq. (C1) (d)	Replace $-p_t - v p_y$ by $-P_t - v P_y$
156	Eq. (C2)	Replace $(\rho_t + \rho_y)$ by $(\rho_t + v \rho_y)$
157	9	Should read $\|v u_y\| << \|u_t\|$ $\|v v_y\| << \|v_t\|$
	Eq. (C3) (b)	Replace p_y by P_y
161	6*	Replace Vol. II by Vol. I
163	1*	Should read: $= \left\{w\,[u_x - s] - u\,[w_x + \sigma]\right\}_x - (wu)_t - \ldots$
171	Eq. (D39a)	Replace $\frac{1}{1+\sigma}$ by $\frac{\sigma}{1+\sigma}$

Page	Line	Correction
171	Eq. (D39b)	Replace $\frac{\sigma}{(1+\sigma)^2}$ by $\left(\frac{\sigma}{1+\sigma}\right)^2$
	Eq. (D40a)	Replace $\frac{1}{1+\sigma}$ by $\frac{\sigma}{1+\sigma}$
	Eq. (D40b)	Replace $\frac{\sigma}{1+\sigma}$ by $\left(\frac{\sigma}{1+\sigma}\right)^2$
173	Eq. (D44)	Replace $-M_2 \sqrt{\frac{k^2}{b}}$ by $-M_2 \gamma \sqrt{\frac{k^2}{b}}$
183	Eq. (D81b)	Replace $\widetilde{DIV}\ \tilde{u}$ by $\widetilde{DIV}\ \vec{\tilde{q}}$
185	Eq. (D90)	Insert brackets in 2nd line so that $\frac{1}{k^2}$ multiplies both terms
	Eq. (D91)	Replace $\frac{1}{\sqrt{2\pi k^2}}$ by $\frac{1}{\sqrt{2\pi}\, k^2}$ in both lines
	1*	Replace $-\beta^2$ by $+\beta^2$ in first matrix
188	1	Should read: $$\int_{-\infty}^{\infty} e^{i\beta x}\, \frac{e^{-\sqrt{\beta^2+\frac{U^2}{4\nu^2}}\,\lvert y\rvert}}{\sqrt{\beta^2+\frac{U^2}{4\nu^2}}}\, d\beta$$
	Eq. (D98a)	Replace $\frac{1}{2\pi U}$ by $\frac{1}{2\pi\nu}$
	2*	Should read: $$\gamma\sqrt{\frac{k^2}{a}} = \frac{1}{2\sqrt{\frac{k^2}{a}+\beta^2}}\, e^{-\lvert y\rvert\sqrt{\frac{k^2}{a}+\beta^2}} = \dots$$
189	Eq. (D101)	Replace $\frac{1}{2\beta\sqrt{1-M^2}}$ by $\frac{1}{2\lvert\beta\rvert\sqrt{1-M^2}}$
	8*	Multiply both integrals by i: $$\Gamma_{c_{11}} \sim \frac{i}{4\pi U}\int_{-\infty}^{\infty} \dots$$
	7*	Add: $$+\frac{1}{2\pi U}\int_0^{\infty} e^{-\lvert y\rvert\beta} \sin\beta x\, d\beta$$

Page	Line	Correction
189	4*	Replace $-e^{\frac{Ux}{\nu}}$ by $-e^{\frac{Ux}{2\nu}}$
	1*	Replace $\frac{x/\sqrt{1-M^2}}{x^2+(1-M^2)y}$ by $\frac{x/\sqrt{1-M^2}}{x^2+(1-M^2)y^2}$
195	12	Add: γ = Euler's constant = 0.577 = 0.577
	9*	Add: $\lambda = \frac{U}{2\nu}$

APPENDIX E

Further Discussion of Flow Past a Flat Plate

In Ref. 61 Lewis and Carrier found the solution of the Oseen equations for stationary incompressible flow past a semi-infinite flat plate at zero angle of attack. In the light of this solution various ideas discussed in this report may be clarified. Lewis and Carrier obtained their solution with the aid of the Wiener-Hopf technique (Cf. Chapter 3). Their solution will be verified below and discussed from the points of view presented earlier in this report. Some questions regarding the validity and nature of boundary layer theory will be discussed in detail for this specific example. The compressible case and the case of a finite plate will also be considered.

Ea. Additional Comments on Splitting

In § 1.5 it was shown that any solution of the linearized equations for a viscous compressible fluid splits into a longitudinal and a transversal part. It is sometimes convenient to split the transversal part further. We shall restrict ourselves to the two-dimensional stationary case, compressible or incompressible. The transversal wave, $\vec{q}_T$, due to a singular shearing force, directed along the x-axis, splits into two parts (Cf. p. 87)

$$\vec{q}_T = \vec{q}_2 + \vec{q}_2^{\,*} \tag{E1a}$$

$$\vec{q}_2 = (u_2, 0) \tag{E1b}$$

$$\vec{q}_2^{\,*} = (u_2^*, v_2^*) = -\frac{1}{2\lambda} \text{ GRAD } u_2 \ , \ \lambda = \frac{U}{2\nu} \tag{E1c}$$

The signifance of this splitting is discussed on pp. 88-90. Actually, any transversal wave may be split in a similar manner. For example, see Lamb (Ref. 16) for incompressible flow; cf. also Ref. 62. This general splitting can be seen as follows. Consider a function $q(x,y)$ satisfying

$$\Delta q = 2\lambda q_x \tag{E2}$$

Now, construct a vector field $\vec{q_T}$ according to (E1), by using the function q for u_2. Obviously the components of $\vec{q_2^*}$ also satisfy (E2). It is also immediately verified that $\vec{q_T} = \vec{q_2} + \vec{q_2^*}$ satisfies the continuity equation. Hence $\vec{q_T}$ is a transversal wave according to (1.57). The force $\vec{X}$ in (1.57) is zero in the field. Conversly, given any solution of (1.57) with $\vec{X} = 0$ one may find the splitting. Since $v_T = v_2^*$, the appropriate u_2 is given by the formula $u_2 = 2\lambda . \int_y^\infty v_2^* \, dy$, and from this u_2, $\vec{q_2^*}$ is constructed according to (E1c). For flow past flat plates at zero angle of attack the component u_2 will be closely related to the boundary layer (cf. p. 90). This, however, is not true in general.

Note that v_2^* is proportional to the vorticity: $\omega = -\dfrac{\partial u_2}{\partial y} = 2\lambda v_2^*$.

Eb. General Relations for Flat Plate

Consider a flat plate of zero thickness at zero angle of attack along the x-axis between $x = a$ and $x = b$ (where b may be $+\infty$). Assume the perturbation velocity in the x-direction to be a constant u_0 along the plate. For no-slip $u_0 = -U$. Then the boundary conditions and conditions of symmetry are (cf. § 3.1 and Eq. (3.45))

$$u(x, 0) = u_0 \qquad (a < x < b) \tag{E3a}$$

$$u_y(x, 0) = 0 \qquad (x < a,\ x > b) \tag{E3b}$$

$$v(x, 0) = 0 \tag{E3c}$$

$$u(-\infty, y) = v(-\infty, y) = 0 \tag{E3d}$$

$$u(x, \infty) = v(x, \infty) = 0 \tag{E3e}$$

The following relations are easily proved for the general compressible case. Let $\vec{q}_1 = (u_1, v_1)$ be the longitudinal wave and let $\vec{q} = \vec{q}_1 + \vec{q}_T$. Since $v = 0$ on the x-axis $v_1 = -v_2^*$ on the x-axis. Also, since $\vec{q}_1$ and $\vec{q}_2^{\,*}$ are irrotational, $\frac{\partial(u_1 + u_2^*)}{\partial y} = \frac{\partial(v_1 + v_2^*)}{\partial x}$ and the right hand side is zero on the x-axis. Hence $\frac{\partial u}{\partial y} = \frac{\partial u_2}{\partial y} = -\omega$ along the x-axis. This means that the skin friction comes only from the u_2 component. By (Elc) the skin friction is then proportional to v_2^* on the plate and hence also to v_1. Off the plate but on the x-axis the skin friction must be zero; hence v_2^* vanishes and consequently also v_1, $\frac{\partial u_2^*}{\partial y}$ and $\frac{\partial u_1}{\partial y}$. Summarizing

On the plate $(a < x < b,\ y = 0)$:

$$\frac{\partial u}{\partial y} = \frac{\partial u_2}{\partial y} = -\omega = -2\lambda v_2^* = 2\lambda v_1$$

$$= \frac{1}{2\mu}\ \text{(skin friction / unit length)} \tag{E4a}$$

$$v_1 = -v_2^*\ , \quad \frac{\partial u_1}{\partial y} = -\frac{\partial u_2^*}{\partial y} \tag{E4b}$$

$$u = u_0 = u_1 + \left(1 - \frac{1}{2\lambda}\frac{\partial}{\partial x}\right) u_2 \tag{E4c}$$

Off the plate $(x < a$ or $x > b,\ y = 0)$:

$$v_1 = -v_2^* = 0, \quad \frac{\partial u_1}{\partial y} = \frac{\partial u_2^*}{\partial y} = \frac{\partial u_2}{\partial y} = 0\ , \quad \omega = 0 \tag{E4d}$$

Note that if u_2 alone is equal to the constant u_0 on the plate, then u_2^* is zero there. Hence by (E4c) the longitudinal wave has to be such that $u_1 = 0$ on the plate. In general, when u_2 is known the complete flow field is easily constructed. $\overrightarrow{q_2^*}$ is found according to (E1c). Then v_1 is known for $y = 0$ from which the longitudinal wave is directly found. In the incompressible case the longitudinal wave is most easily found by solving Laplace's equation (cf. Ec, below). In general, u_2 determines the force on the plate by (E4a). Hence the complete flow field may also be obtained by a superposition of fundamental solutions (cf. p. 85 ff) with known source strength.

Ec. Semi-Infinite Plate in Incompressible Flow

For the semi-infinite flat plate the boundary conditions are given by (E3) with $a = 0$ and $b = +\infty$. Furthermore the Oseen equations for incompressible flow are assumed. This is the problem that was solved in Ref. 61. This solution will be verified below and presented in a form suitable for discussion of the boundary layer theory. In the light of the Lewis-Carrier solution it is reasonable to assume that the appropriate u_2 is the parabolic boundary layer solution discussed on pp. 110-114. The flow field corresponding to this u_2 field will be constructed and the boundary conditions (E3) verified. For convenience some properties of parabolic coordinates (ξ , η) will be given here.

$$z = x + iy\,,\quad \zeta = \xi + i\eta\,,\quad z = \zeta^2,\quad x = \xi^2 - \eta^2\,,\quad y = 2\xi\eta \tag{E5}$$

$$2\xi^2 = \left(\sqrt{x^2 + y^2} + x\right)\,,\quad 2\eta^2 = \left(\sqrt{x^2 + y^2} - x\right)$$

$$\frac{\partial\eta}{\partial x} = -\frac{1}{2}\,\frac{\eta}{\xi^2 + \eta^2}\,,\quad \frac{\partial\eta}{\partial y} = \frac{1}{2}\,\frac{\xi}{\xi^2 + \eta^2}$$

The x,y-plane is slit along the positive x-axis and $\sqrt{z}$ is positive for $y = 0$ on the top of the slit. Hence η is always positive, ξ

has the same sign as y . $\eta = 0$ is the plate, $\xi = 0$ is the x-axis upstream of the plate.

Now we put

$$u_2 = u_0 \ \text{ERFC} \ (\eta \sqrt{2\lambda}) \tag{E6}$$

which is Eq. (3.37) of the report.

According to (E1c)

$$\overrightarrow{q_2^*} = -\frac{u_0}{2\lambda} \ \text{GRAD} \ (\text{ERFC} \ \eta \ \sqrt{2\lambda}) \tag{E7}$$

This shows that $\overrightarrow{q_2^*}$ is always perpendicular to the parabolas η = constant and directed towards the plate for $u_0 < 0$. Its u and v components are

$$u_2^* = -\frac{1}{2\lambda} \frac{\partial u_2}{\partial x} = -\frac{u_0}{\sqrt{2\pi\lambda}} \frac{\eta}{\xi^2+\eta^2} e^{-2\lambda\eta^2} \tag{E7'a}$$

$$v_2^* = -\frac{1}{2\lambda} \frac{\partial u_2}{\partial y} = \frac{u_0}{\sqrt{2\pi\lambda}} \frac{\xi}{\xi^2+\eta^2} e^{-2\lambda\eta^2} \tag{E7'b}$$

For $y = 0$ (E7) reduces to

$$x < 0: \quad u_2^* = -\frac{u_0}{\sqrt{-2\pi\lambda x}} e^{2\lambda x} \quad , \quad v_2^* = 0 \tag{E8a}$$

$$x > 0: \quad u_2^* = 0 \quad , \quad v_2^* = \frac{u_0}{\sqrt{2\pi\lambda x}} \tag{E8b}$$

By the sign convention discussed after (E5), $\sqrt{2\pi\lambda x}$ is positive on the top side of the plate and negative on the lower side.

A longitudinal wave $\overrightarrow{q_1}$ has to be added to $\overrightarrow{q_1^*}$ to make $v = 0$ for $y = 0$. In the incompressible case $\overrightarrow{q_1} = \text{GRAD} \ \varphi$ where $\Delta\varphi = 0$. This means that $u_1 - i v_1$ is an analytic function of $z = x + iy$. The boundary condition is that on the x-axis $v_1 = v_2^*$ where v_2^* is given by (E8). This problem has a solution which is easily found, namely,

$$u_1 - iv_1 = \frac{iu_o}{\sqrt{2\pi\lambda z}} = \frac{u_o}{\sqrt{2\pi\lambda}}\left(\frac{\eta}{\xi^2+\eta^2} + i\,\frac{\xi}{\xi^2+\eta^2}\right) \tag{E9}$$

For $y = 0$ (E9) reduces to

$$x < 0: \quad u_1 = \frac{u_o}{\sqrt{-2\pi\lambda x}}, \quad v_1 = 0 \tag{E10a}$$

$$x > 0: \quad u_1 = 0, \quad v_1 = -\frac{u_o}{\sqrt{2\pi\lambda x}} \tag{E10b}$$

The longitudinal wave $\vec{q_1}$ was found from the condition on v. If u_1 were also prescribed the problem would be over determined. However, fortunately the correct u_1 value is obtained. According to (E10b) u_1 is actually zero on the plate. Further u_2^* is also zero according to (E8b) and u_2 was constructed so as to give the value u_o on the plate (cf. pp. 110-114). Hence on the plate $u = u_1 + u_2^* + u_2 = u_2 = u_o$. Thus the boundary condition (E3a) is verified. The condition $v = 0$ for $y = 0$ is fulfilled by construction and the other conditions in (E3) are easily checked. Hence the solution to the problem of the semi-infinite flat plate has been found. It is

$$\vec{q} = \vec{q_1} + \vec{q_2^*} + \vec{q_2} \tag{E11}$$

where $\vec{q_2} = (u_2, 0)$ and u_2 given by (E6), $\vec{q_2^*}$ is given by (E7) and $\vec{q_1}$ by (E9) (cf. also (E20)). In Ref. 61 the solution is given in form of integrals. It is, however, equivalent to (E11). In particular it will be verified below that the two forms of the solution give the same values for the skin friction.

Various other flow quantities, such as vorticity, stream function, pressure and skin friction may now be obtained from (E11).

These formulas will be listed below.

The vorticity ω is carried by u_2 alone since $\vec{q_1}$ and $\vec{q_2^*}$ are gradients. The vorticity is

$$\omega = -\frac{\partial u_2}{\partial y} = 2\lambda v_2^* = u_o\sqrt{\frac{2\lambda}{\pi}}\ \frac{\xi}{\xi^2+\eta^2}\ e^{-2\lambda\eta^2} \tag{E12}$$

The stream function ψ_T for the transversal wave may be obtained as $\psi_T = -\int_{-\infty}^{x} v_2^*\, dx = \frac{1}{2\lambda}\frac{\partial}{\partial y}\int_{-\infty}^{x} u_2\, dx$. The integration is straightforward but the resulting expression is rather awkward and does not seem particularly useful. It is interesting to note how u_2 is somewhat similar to a stream function. The velocity components of the transversal wave are found from u_2 by differentiation and addition in such a manner that validity of the equation $\mathrm{DIV}\ \vec{q_T} = 0$ is automatically assured.

$$\psi_T = \frac{u_o}{2\lambda}\frac{\partial}{\partial y}\left\{ \mathrm{ERFC}\ (\eta\sqrt{2\lambda})\ \left[x+\lambda y^2+\frac{1}{4\lambda}\right] + \frac{2}{\sqrt{\pi}}e^{-2\lambda\eta^2}\left[\frac{y^2}{\eta}\sqrt{\frac{\lambda}{8}} - \frac{\eta}{\sqrt{8\lambda}}\right]\right\} \tag{E13a}$$

The stream function ψ_L for the longitudinal wave is immediately obtained as

$$\psi_L = Re\left(u_o\sqrt{\frac{2z}{\pi\lambda}}\right) \tag{E13b}$$

In the incompressible case, the pressure perturbation is obtained from u_1 according to the linearized Bernoulli's law

$$p - p_\infty = -\rho\, U u_1 = -\frac{\rho U u_o}{\sqrt{2\pi\lambda}}\ \frac{\eta}{\xi^2+\eta^2} \tag{E14}$$

According to (E10) $p = p_\infty$ on the plate. On the negative x-axis it is $\frac{-\rho U u_o}{\sqrt{-2\pi\lambda x}}$

The skin friction is (cf. (E4a) and (E8b))

Skin friction per unit length $= 2\mu \dfrac{\partial u}{\partial y} = 2\mu \dfrac{\partial u_2}{\partial y}$

$$-2\mu\omega = -4\mu\lambda\,|v_2^*| = -\frac{2\rho U u_o}{\sqrt{2\pi\lambda x}} \tag{E15}$$

(E15) is identical with the formula for skin friction obtained in a different way in Ref. 61.

Ed. Comparison with Boundary Layer Theory. Skin Friction and Displacement Thickness

The solution presented in Ec shows several interesting features in regard to boundary layer theory. The boundary layer which is relevant to the discussion is the linearized boundary layer. See, for example, p. 34-35 and 109-110 where the boundary layer approximations are carried out on the Oseen equations. The x-momentum equation becomes the heat equation (1.59) and has the solution for the u-component, u_{BL} (3.32)

$$u_{BL} = u_o \,\mathrm{ERFC}\left(\frac{\lambda y^2}{2x}\right)^{1/2} \tag{E16}$$

Another way of obtaining (1.59) is to linearize the Prandtl-Blasius boundary layer equation. The solution is due to Rayleigh who obtained it as a non-stationary solution and identified the time with x/U. However, the first method is the most relevant for our purposes. The Oseen equations are regarded as a model for rather than an approximation to the Navier-Stokes equations. As pointed out in the Introduction and in § 1.7, one important aim is to study in detail various difficulties of the Navier-Stokes equation for this simplified model.

One of the purposes of boundary layer theory is to predict the skin friction. A comparison of (E16) and (E15) shows the following surprising result: For the linearized equations, solutions of the full equation and of the boundary layer equation give exactly the same skin friction for the semi-infinite plate including the region near the nose of the plate. Incidentally, this is also the skin friction given by the parabolic boundary layer (u_2). In formulas

$$\frac{\partial u}{\partial y} = \frac{\partial u_2}{\partial y} = \frac{\partial u_{BL}}{\partial y} \qquad \text{for } y=0 \tag{E17}$$

Another idea of classical boundary layer theory is that the flat plate has an apparent thickness due to the boundary layer. The external, that is non-viscous, flow around a body with thickness may then be computed. This would give an external pressure gradient, which might be used for recomputing the boundary layer etc. (cf. Ref. 63). The notion of apparent thickness is made precise by the introduction of the notion of displacement thickness, $\delta^*(x)$. Its definition, (motivated by the continuity equation) and its value in our special case is

$$\delta^*(x) = -\frac{1}{U}\int_0^\infty u_{BL}\, dy = -\frac{2u_0}{U}\sqrt{\frac{x}{2\pi\lambda}} \tag{E18}$$

The apparent shape of the body is thus parabolic. Various interpretations of this notion will now be discussed:

1. The customary procedure is to compute the potential flow outside the parabolic body defined by (E18). That is, this potential flow exists only for $x<0$ and for $x>0$, $|y| > \delta^*(x)$. It is patched onto the boundary layer solution along the parabola $y=\delta^*(x)$. This new external flow would give a pressure gradient along this parabola. With the aid of this pressure gradient the boundary layer may be recomputed. However, no matter how far the iteration is continued, the

flow will be patched along the edge of the boundary layer. Inside the boundary layer, the customary approximations will be made, i.e. p_y will always be zero.

2. A second procedure is to satisfy the boundary condition (condition of tangency) required by the parabolic body on the x-axis, in the manner familiar from linearized non-viscous theory. This would give

$$\begin{aligned} v &= U \frac{d(\delta^*(x))}{dx} = -\frac{u_o}{\sqrt{2\pi\lambda x}} && \text{for} \quad x>0,\ y=0 \\ v &= 0 && \text{for} \quad x<0,\ y=0 \end{aligned} \tag{E19}$$

But the problem for linearized non-viscous flow defined by (E19) is identical with the problem of finding the longitudinal component q_1 of the complete solution. Hence this interpretation yields the correct longitudinal wave, in particular the correct pressure field. Note that this longitudinal wave exists inside the boundary layer. In particular, p_y is not zero in the region of the boundary layer, not even for $y=0$. On the other hand $p_x = 0$ on the plate so that the iteration procedure described above would terminate. Note also that in this procedure the use of the displacement thickness amounts to determining the value of the v-component at the plate from an integral of the continuity equation. This gives $v(x,0) = -\int_0^\infty \frac{\partial(u_{BL})}{\partial x} dy$ which is seen to be equal to $\frac{U d(\delta^*(x))}{dx}$. On the other hand, u_{BL} plus the longitudinal wave does not satisfy the continuity equation.

3. These ideas should be compared with the discussion of the bulging of the boundary layer discussed previously in § 2.7 and § 3.3. Here the starting point was u_2 rather than u_{BL}. From u_2 and the continuity equation a v-component was determined which will be denoted

by v_2 here. (u_2, v_2) is then a transversal wave. This v_2 is not equal to zero for $y=0$, even for $x<0$. Then a longitudinal wave was added to satisfy the condition $v=0$ for $y=0$. The sum of the longitudinal wave and the transversal wave then satisfies the complete flow equations and the condition $v=0$ for $y=0$ but violates the conditions on u and u_y on the x-axis.

The three methods will now be compared. The first two methods are similar and only the second one has to be discussed. It gives better results than the first one, whose accuracy would even decrease if iteration were used. An interesting aspect of method two is that it gives the correct longitudinal wave in spite of the fact that it is based on u_{BL} rather than u_2. On the other hand it gives very little information about the transversal wave except at (the very important). region close to the plate. In particular it indicates no upstream spreading of vorticity. Its results may not be improved by iteration.

The third method yields the correct vorticity and a better approximation to the transversal wave. On the other hand the longitudinal wave is only correct asymptotically far downstream where u_{BL} approximates u_2 and $\vec{q}_2^{\,*}$ becomes less important. Note that the asymptotic formula (3.313) agrees with (E19). On the other hand, the third method may be iterated and will presumably give increasingly better results.

All iteration methods are based on the idea of satisfying the continuity equation and the condition $v=0$ at $y=0$. This may be compared with the splitting discussed above. If one starts with

$\vec{q}_1$, then $\vec{q}_2^{\,*}$ may be added to satisfy the continuity equation. Finally the longitudinal component has to be added to satisfy the boundary condition on v . However, in general it is of course difficult to find the true u_2-component, whereas u_{BL} is easy to find.

Above it has been shown that boundary layer gives the exact skin friction and the exact pressure field for a semi-infinite plate. Later it will be shown that this is only approximately correct for a finite plate or when compressibility is taken into account. Presumably it is also only approximately true in the non-linear case. From the linearized point of view it is easy to verify that Prandtl's physical ideas about boundary layer theory are correct (orders of magnitude of various quantities in the boundary layer may also easily be checked from the solutions given). A different aspect of boundary layer theory will be considered in the following section.

Ee. Boundary Layer Theory as First Step in an Exact Expansion

In § 2.7, § 3.3 and in Ef boundary layer theory was regarded as the first step in some rather loosely defined iteration procedure. The question of making this iteration exact will now be discussed. In particular one may ask: Is it possible to expand the complete solution of the Oseen equation in some power series such that the boundary layer theory gives the first term in the expansion, and in general the n^{th} term is obtained from the preceding terms? The n^{th} term would have to satisfy an appropriate differential equation which together with its boundary conditions may involve the lower order terms as known functions.

In order to study this point consider the solutions obtained previously

$$u = u_1 + u_2 + u_2^* , \quad u_2 = u_0 \, \mathrm{ERFC}\, (\eta \sqrt{2\lambda}) \tag{E20a}$$

$$u_1 + u_2^* = \frac{u_0}{\sqrt{2\pi\lambda}} \frac{\eta}{\xi^2+\eta^2} \left[1 - e^{-2\lambda\eta^2}\right] \tag{E20b}$$

$$v = v_1 + v_2^* = -\frac{u_0}{\sqrt{2\pi\lambda}} \frac{\xi}{\xi^2+\eta^2} \left[1 - e^{-2\lambda\eta^2}\right] \tag{E20c}$$

As $1/\lambda$ $(\sim \nu)$ tends to zero this solution behaves properly. It tends to the non-viscous solution $u = v = 0$, except on the plate ($\eta = 0$) where $u = u_0$. However it is apparent that in the neighborhood of the plate the convergence is non-uniform. This phenomenon is typical of singular perturbation problems (§ 1.7).

The main problem is the behavior of the solution for small values of $1/\lambda$ or ν. If one could develop u and v in power series of $1/\lambda$ around $1/\lambda = 0$, then the coefficient of $1/\lambda$ in this series would be regarded as the perturbation of the non-viscous solution for small values of ν. However, regarding ξ and η (or x and y) as parameters it is immediately obvious that u and v have a singular dependence on ν at $\nu = 0$. The term $e^{-\frac{\bar{U}\eta^2}{\nu}}$, regarded as a function of ν has a complicated singularity at $\nu = 0$. The function itself and all its derivatives are zero at $\nu = 0$. Hence it may not be developed in power series around this point. In the theory of a complex variable one expresses the equivalent fact by saying that e^z has an essential singularity at $z = \infty$. A similar statement is true for the error function. The situation is not remedied by introducing $\nu^{1/2}$ or any other fractional power as a new variable (e^z is single valued at $z = \infty$). Thus this simple type of perturbation procedure is completely useless in the present case. These complications are typical of a singular perturbation procedure. The type of singularity encountered above occurs frequently in singular perturbation problems. An instructive example is discussed by Friedrichs in Ref. 52, p. 71, ff: another example occurring in hydrodynamics is studied in Ref. 83. As a contrast, compare the interesting example of non-analytic dependence on a parameter in a regular perturbation problem given by Oseen (Ref. 21, pp. 18-20). In this example, suggested by hydrodynamical equations, the solution is a power series in the absolute value of the parameter.

Now it might be thought that by a different choice of variables the singularity could be eliminated and the solution given a reasonable expansion. This can be done and improves the situation to some extent, as will be shown. It is clear from (E20) that if η is combined with $\sqrt{\lambda}$ the singularity in λ may be removed. That is, u and v regarded as functions of ξ, $\eta\sqrt{\lambda}$, and $1/\sqrt{\lambda}$, are analytic functions for all values of the last argument including zero. u_2 and the expressions in brackets in (E20a) and (E20b) are thus independent of $1/\sqrt{\lambda}$. Power series development in $1/\sqrt{\lambda}$ depends then only on the factors $\frac{1}{\sqrt{\lambda}}\,\frac{\eta}{\xi^2+\eta^2}$ and $\frac{1}{\sqrt{\lambda}}\,\frac{\xi}{\xi^2+\eta^2}$. These may be written as

$$\frac{1}{\lambda}\,\frac{\eta\sqrt{\lambda}}{\xi^2\left(1+\frac{\eta^2\lambda}{\xi^2}\cdot\frac{1}{\lambda}\right)} = \frac{\eta\sqrt{\lambda}}{\xi^2}\left(\left(\frac{1}{\sqrt{\lambda}}\right)^2 - \left(\frac{\eta^2\lambda}{\xi^2}\right)\left(\frac{1}{\sqrt{\lambda}}\right)^4 + \cdots\right) \tag{E21a}$$

$$\frac{1}{\sqrt{\lambda}}\,\frac{\xi}{\xi^2\left(1+\frac{\eta^2\lambda}{\xi^2}\cdot\frac{1}{\lambda}\right)} = \frac{1}{\xi}\left(\frac{1}{\sqrt{\lambda}} - \left(\frac{\eta^2\lambda}{\xi^2}\right)\left(\frac{1}{\sqrt{\lambda}}\right)^2 + \cdots\right) \tag{E21b}$$

In such an expansion u_2 is of order unity, v begins with the term $1/\sqrt{\lambda}$ and $u_1 + u_2^*$ with the term $(1/\sqrt{\lambda})^2$. However, it is seen that this expansion converges only for $\frac{\eta^2 \lambda}{\xi^2} \cdot \frac{1}{\lambda} < 1$, that is $\eta < |\xi|$ or $x > 0$. The choice of variables made and the expansion in $1/\sqrt{\lambda}$ is very close to boundary layer theory. The first terms in this expansion are the most important for $\eta << |\xi|$. The first term in the expansion of u is u_2 which for $\eta << |\xi|$ or $\frac{|y|}{\sqrt{x}} << 1$ is approximately u_{BL} (cf. (E16) and p. 112). v starts out with a term of order $1/\sqrt{\lambda}$ which is thus of higher order than u_2 but of lower order than the correction to u_2 : $u_1 + u_2^*$.

Up to this point the results are a further confirmation of Prandtl's boundary layer theory which could be followed up in detail by checking the order of magnitude of the various terms u_{xx}, u_{yy} etc. in the Oseen equations. However, the main problem of interest in this section is: Is it possible to expand u and v as

$$u = \sum_{i=0}^{i=\infty} f_i\,(\xi, \eta\sqrt{\lambda})\left(\frac{1}{\sqrt{\lambda}}\right)^i , \quad v = \sum_{i=0}^{i=\infty} g_i\,(\xi, \eta\sqrt{\lambda})\left(\frac{1}{\sqrt{\lambda}}\right)^i \qquad \text{(E22)}$$

and then determine the differential equations and boundary conditions which f_i and g_i would have to satisfy? We have already seen that such an expansion fails completely if f_i and g_i are required to be functions of ξ and η. Even with the improved variable $\eta\sqrt{\lambda}$ the answer is negative. From (E20) and (E21) it follows that the expansion (E22) exists for $x > 0$ but diverges for $x < 0$. In the first region the lowest order term would satisfy the boundary layer equations. Formally one may obtain equations for the higher order terms. However, even this method cannot give the correct flow field since (E22) is divergent for $x < 0$. For $x < 0$, the expansion procedure would give an expansion in powers of λ rather than its inverse. From the point of view of dimensional analysis, u/u_0 is a function of only two variables such as $\xi\sqrt{\lambda}$, $\eta\sqrt{\lambda}$ or $x\lambda$ and $y\lambda$. Thus an expansion depends actually only on these local Reynolds numbers. This is implicit in the statement above that the boundary layer approximation is good for $\eta \ll |\xi|$, which is equivalent to $\eta\sqrt{\lambda} \ll |\xi\sqrt{\lambda}|$.

In Ref. 68 an expansion of the type (E22) (actually an equivalent expansion for the stream function) was assumed, based on some remarks in Ref. 67 (p. 565). The author studied the non-linear Navier-Stokes equations rather than the linearized Oseen equations. A second-order approximation was found which, however, showed some pathological features such as an non-integrable singularity of the skin friction at the nose. Perhaps this is connected with the non-existence of the expansion (E22) for $x < 0$.

Summarizing, it may be said that the original claims of Prandtl's boundary layer theory are verified from various points of view, in the linearized solution. On the other hand it is not possible in the linearized case to regard the boundary layer as giving the first term of a power series expansion which would describe the flow field everywhere. It is almost certainly also not possible in the non-linear case. There still remains the possibility that boundary layer theory is the first step in some expansion of asymptotic nature or in some "patching" procedure. These possibilities will not be discussed here. However, the explicit solution (E20) should furnish a good method of testing any such procedure.

Ef. Semi-Infinite Flat Plate: Compressible Case

A brief discussion will now be given about the effects of compressibility in the problem of the semi-infinite plate. For the discussion the Oseen equations are replaced by the more general system (1.43), with the force $\vec{X} = 0$. In this application to the flat plate problem the linearized equations are more unrealistic for high Mach numbers than they were for the incompressible case. For high Mach numbers dissipation of energy, conduction of heat, and variation of Mach number all become important in the boundary layer and these effects are neglected in the linearized equations. However, these equations are useful as a model in illustrating the relationship of the outer flow and the boundary layer.

The boundary layer equations of (1.43) are the same as for the incompressible case. Thus according to boundary layer theory the skin friction would be given by (E15). That this result is not correct for the general compressible case can be seen as follows: Consider the flow field arising from the external force distribution given by boundary layer theory. It consists of the flow field for the incompressible case (E20) plus a field arising from the compressibility correction to the fundamental solution (p. 93 ff). The boundary layer solution for the skin friction would be correct only if u_c, the u component of the compressibility correction, were zero on the plate. This, however, is not true. The value of u_c is

$$u_c(x,0) = \frac{u_0}{\sqrt{\pi\lambda}} \int_{\frac{3\lambda}{2M^2}(1-M^2)}^{\frac{3\lambda}{2M^2}} \frac{e^{-x\sigma}}{\sqrt{\sigma}} \sqrt{\frac{\frac{3\lambda}{2M^2} - \sigma}{\sigma - \frac{3\lambda}{2M^2}(1-M^2)}}\, d\sigma \qquad x>0,\ M<1 \tag{E23a}$$

$$u_C(x,0) = \frac{u_o}{\sqrt{\pi\lambda}} \int_0^{\frac{3\lambda}{2M^2}} \frac{e^{-x\sigma}}{\sqrt{\sigma}} \sqrt{\frac{\frac{3\lambda}{2M^2} - \sigma}{\sigma + \frac{3\lambda}{2M^2}(M^2-1)}}\, d\sigma \qquad x>0,\ M>1 \qquad \text{(E23b)}$$

As $x \rightarrow \infty$

$$u_c \sim \frac{u_o M e^{-\frac{3\lambda}{2M^2}(1-M^2)x}}{\sqrt{1-M^2}\sqrt{\lambda x}} \qquad M<1 \qquad \text{(E24a)}$$

$$\sim \frac{u_o}{\sqrt{M^2-1}\sqrt{\lambda x}} \qquad M>1 \qquad \text{(E24b)}$$

The case $M = 1$ must be treated separately.

It is thus plausible that boundary layer theory gives a good approximation far downstream. However, near the leading edge the flow field and the skin friction are given incorrectly. The magnitudes of the errors depend on the Mach number M .

The exact solution for arbitrary Mach number cannot be given at present. However, one interesting limiting case may be treated. It can be seen from (E23) that as $M \rightarrow \infty$, $u_c \rightarrow 0$ on the plate. Hence in this case boundary layer theory again gives the correct skin friction from the leading edge to infinity. This result could also be derived by a comparison of the fundamental solutions for the velocity field at zero and infinite Mach numbers. It now follows that the transversal part of the flow field is the same for $M = 0$ as for $M \rightarrow \infty$. However, in the latter case the potential of the longitudinal waves no longer satisfies Laplace's equation but satisfies instead the equation derived from (1.56e) by letting $M \rightarrow \infty$:

$$\frac{4\nu}{3}\, \Delta\, \Phi_x - U\, \Phi_{xx} = 0 \qquad \text{(E25)}$$

This equation is identical with the dynamical equation for transverse waves with ν replaced by $\frac{4\nu}{3}$. Thus the longitudinal wave, to within a multiplicative constant, is given by $\vec{q}_2^{\,*}$ with ν replaced by $\frac{4\nu}{3}$. The pressure for this case is no longer found from the linearized Bernoulli law but can be found from the condensation vector with $M \to \infty$ (p. 183 ff). The pressure perturbation in this case is not zero but constant on the plate and it has the value:

$$\frac{p - p_\infty}{p_\infty} = \frac{\sqrt{3}}{2} \tag{E26}$$

As mentioned before, although this solution may not be considered a description of hypersonic flow of a real fluid past a flat plate, it still has interest as a model for this flow. For hypersonic Mach numbers the idea of a boundary layer and a separate outer flow has to be revised, for the following reason: The Mach waves of the non-viscous flow lie very close to the plate for high Mach numbers, parallel to the plate at infinite Mach number. If, as before, the longitudinal part of the flow is considered as a disturbance diffusing about the sub-characteristic, it is clear that most of this wave is very close to the plate. Thus, in a sense, the longitudinal wave is within the boundary layer. The entire flow field must be considered.

This is what has been done above and the results are quite interesting. It was shown that the linearized boundary layer theory (i.e. Rayleigh's solution) gives the correct skin friction all along the plate. However, the velocity profile is nowhere given correctly, not even far downstream. This happens because the longitudinal part of the solution does not vanish far downstream. In addition, one of the usual assumptions of boundary layer theory is violated, namely the assumption that the pressure is constant across the boundary layer. The pressure on the plate is given by (E26) and the pressure perturbation drops to zero quite rapidly on going away from the plate. The usefulness of boundary layer theory in this case is somewhat paradoxical.

It should be remarked that the skin friction depends on the value of ν which is assumed. In the linearized theory this value is supposed to be taken in the free stream. Thus, although the form of the skin friction is identical for zero and for infinite Mach number the actual values predicted by this theory will probably differ in any given case.

Eg. Flat Plate of Finite Length in Incompressible Flow

In this section a brief discussion will be given of stationary incompressible flow past a finite flat plate at zero angle of attack. The boundary conditions are given in § Eb. The coordinates of the leading and trailing edges will be assumed to be $(-b, 0)$ and $(b, 0)$ respectively. The general discussion in § Eb, in particular (E4) applies. The Oseen equations are equivalent to the integral equations (3.23). In contrast to the semi-infinite flat plate this problem has an overall Reynolds number based on the length of the plate: $R = 4b\lambda = \dfrac{2Ub}{\nu}$. The nature of the solution will depend on the value of this parameter. Two extreme cases have been studied previously, namely the singular flat plate $(R = 0)$ and the semi-infinite flat plate $(R = \infty)$.

Boundary layer theory predicts the same flow near a finite plate as near the leading part of a semi-infinite plate. This is so because the boundary layer equations are parabolic in x and show no upstream influence. For these linear equations the flow field of the finite plate can be regarded as composed of that due to the semi-infinite plate from the leading edge plus that due to another special semi-infinite flat plate extending downstream from the trailing edge. This second flat plate must have such a distribution of slip that the skin friction due to the first plate is cancelled out. The combination of these two flow fields will be referred to as cutting off a semi-infinite plate. According to boundary layer theory this cutting off cannot spoil the boundary conditions on the remaining part of the plate. On the other hand, according to the full Oseen equations the cutting off must have an upstream effect. In particular, there might be a profound modification of the flow right at the trailing edge. The rest of the plate will be less affected. It can easily be shown that a simple solution like that for the semi-infinite flat plate does not exist. In particular u_1 and u_2^* will not be zero on the plate, and not even constant.

A large number of papers have been written about this problem. A brief survey of this literature follows.

One approach is to use the integral equation (3.23). An exact solution to this equation has not been given. However, various approximations have been carried out. For example, Bairstow, Cave and Lang (Ref. 70) treated numerically an integral equation equivalent to (3.23). They obtained the local and total skin friction for Reynolds numbers of 4 and 4 x 10^4. They found that at the low Reynolds number the local skin friction was almost symmetrical about the mid-chord and had singularities at the leading and trailing edges. At the high Reynolds number the distribution was unsymmetric, the location of the minimum friction moving toward the rear. The singularity at the leading edge was still in evidence and one at the trailing edge was also indicated. By generalizing a solution of Lamb (cf. Ref. 16, p. 606) Bairstow et al obtained the first approximations to the skin friction and drag, which in our notations are

$$f(x) = \frac{4\mu}{\rho} \frac{(+u_0)}{\text{Log}\,\frac{16}{R} - \gamma + 1} \frac{1}{\sqrt{b^2 - x^2}} \qquad \begin{array}{l}\gamma = \text{Euler's const.} \\ = 0.577\end{array} \tag{E27a}$$

$$C_D = \frac{8\pi}{R} \frac{1}{\text{Log}\,\frac{16}{R} - \gamma + 1} \left(\frac{-u_0}{U}\right) \qquad \text{per unit span} \tag{E27b}$$

These solutions should be good for $R \leq 4$.

The distribution of skin friction as given by (E24b) is symmetrical about the mid-point of the plate and there is a square root singularity at both the leading and trailing edges. The effect of the trailing edge is thus felt strongly in the skin friction on the plate. The results (E27) can also be derived by solving the integral equation approximately for low R . If the kernel Γ_{11} is expanded for small λ and only the largest term kept (3.23) becomes

$$u_0 = -\frac{\lambda}{2\pi U} \int_{-b}^{+b} f(\xi) \left\{ \text{Log}\,|x - \xi| + C_3 \right\} d\xi \tag{E28}$$

where $C_3 = \text{Log}\,\frac{\lambda}{2} + \gamma - 1$. This has the solution (E27a). This procedure was in fact carried out by Piercy and Winny (Ref. 76) and they also obtained a second approximation for low R which shows how the distribution of skin friction tends to become unsymmetric as R increases. The skin friction is of the form

$$f(x) = \frac{A + Bx + Cx^2}{\sqrt{b^2 - x^2}} \qquad \text{(E29)}$$

In addition Piercy and Winny considered the high Reynolds number case and solved (3.23) approximately for large values of λ. The asymptotic form of K_0 is substituted in Γ_{11} and in the first approximation no upstream spreading of transversal waves is considered. The integral equation (3.23) thus becomes identical with (3.25) discussed previously and the skin friction is identical with Rayleigh's value. The typical singularity of skin friction near the nose remains but the trailing edge has no effect on the skin friction. The drag coefficient C_D is as usual proportional to $\frac{1}{\sqrt{R}}$. In other words, this solution is simply an application of boundary layer theory. Piercy and Winny also obtain a second approximation, a term in $\frac{1}{R}$, for large R.. They approximately check Bairstow et al's values for low and high R. In any of the approximations above it is difficult to estimate the error which is being made. However, Piercy and Winny show that their two formulas for C_D fit together approximately and give values for C_D over the full range of R. Various other writers have tried similar methods and some of these are listed in the bibliography.

Meksyn (Ref. 80), T. Lewis (Ref. 81) and Davies (Ref. 82) have attacked the problem of the finite plate in another way by using elliptic coordinates. This method leads naturally to a solution in terms of Mathieu functions. In the last paper, which is that of Davies, the analysis is heavy and no essentially new results appear. The drag coefficient at low R is found in the first and second approximations and a small discrepancy is noted between the second approximation of the author and that of Piercy and Winny.

Carrier and Lewis (Ref. 61) indicate, but do not carry out, still another method of solving the problem for the finite plate. In the form suggested their solution seems unsuited for describing the flow near the trailing edge.

In conclusion, in spite of the fact that no exact solution exists for the finite flat plate even in incompressible flow some general statements can be made about the flow field. Consider first the case of low Reynolds number. Then one may use an approximation for the fundamental solution that is valid in a small neighborhood as done in (E25). This approximate kernel is symmetrical and the resulting solution for the force distribution is symmetrical. Near the plate one has approximately symmetrical Stokes type flow. However, at larger distances from the plate even a symmetrical force distribution gives an asymmetric flow field since the other terms of the fundamental solution have to be taken into account. Actually the force distribution is symmetric only in the limit $R \to 0$. As the Reynolds number increases the minimum of the skin friction shifts towards the trailing edge while the singularities at both edges remain. In the high Reynolds number case the values of the fundamental solution both near and far from the point of application will influence the solution for the flat plate. However, using an expansion valid at large distances from the fundamental solution actually gives a good approximation except at the trailing edge. This may be seen from the cutting-off procedure discussed above.

The role of boundary layer theory may be summarized as follows: First of all, boundary layer theory for the finite flat plate has all the limitations of that for the semi-infinite plate. In addition it is a good approximation only for high overall Reynolds number. In this case it is valid locally only some distance downstream from the leading edge. It gives the correct form of the skin friction singularity at the leading edge. In addition, it is not valid near the trailing edge. The skin friction becomes infinite as the square root of the distance from the trailing edge. Boundary layer theory shows no singularity here. The upstream influence of the trailing edge dies out very fast. All distances are of course to be measured non-dimensionally, that is multiplied by λ .

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